Multisymplectic methods in field theory with symmetries and BRST

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ABSTRACT

The generalization of Hamiltonian mechanics to covariant Hamiltonian field theory on multiphase space is given an accessible exposition as a practical method covering various topics, and a classical multiphase space BRST formalism is developed for systems with symmetries and applied to a system with secondary constraints (Yang-Mills).

Contents

Chapter 1

Introduction

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In this thesis I have investigated multisymplectic methods for theories which have symmetries, in particular multisymplectic BRST. This was applied to a system with both primary and secondary constraints using the Hamiltonian as constraint (Yang-Mills), and a sigma model.

In contrast to the familiar Hamiltonian formulation of relativistic field theory on an infinite-dimensional phase space, the finite-dimensional multiphase-space formalism for field theory is the natural covariant Hamiltonian formulation, reflecting the geometry of fields. This thesis summarises the multiphase-space formalism for field theory, and develops a multisymplectic form of classical BRST for systems with gauge symmetry. The multisymplectic formalism is much more restrictive than the symplectic, and the approach in this thesis is to concrete applications to well known physical systems rather than seeking maximum generality or abstraction. Both symplectic and multisymplectic mechanics are presented, in turn, in a fairly didactic and geometrical way, so as to clearly bring out the parallels between the two and to make multisymplectic mechanics more accessible as a generalization of symplectic mechanics. In particular, the original work in this thesis is the generalization of BRST, a modern method of dealing with symmetries, to multisymplectic BRST (in section 4.4) which is applied to a system with secondary constraints (electromagnetic and non-abelian Yang-Mills fields) in section 4.6.4 and the multiphase-space canonical transformations with generating functions in appendix H, and calculating partway the multiphase-space BRST structure of Witten's topological sigma model (chapters 5 and 6).

This thesis has two aims: as a didactic practical introduction to multiphase space field theory and to present some applications, in particular examples of BRST. The first aim is also a lead-in to the second. To help achieve the first aim, classical (phase-space) mechanics is reviewed in chapter 2, covering usual topics (and with the example of electromagnetism), in such a way that a parallel delineation of multiphase space field theory in chapter 3 of the same topics can be compared. Both phase space and multiphase space mechanics are presented most of the time using coordinates rather than more abstract notation, to serve the aims of being accessible and practical. Nevertheless because part of the purpose of the multiphase space formalism is to embody field theory in a 'natural' geometrical form, a fairly large amount of differential geometry, especially of symplectic and multisymplectic manifolds, is employed, even in presenting well known Hamiltonian mechanics in chapter 2. This serves several purposes: that the coordinate presentation can be related to a more general geometrical picture, bringing out the geometrical nature of multiphase space field theory, to relate it to phase space mechanics, and a didactic one of picturing the formalism.

The research aim of this thesis is performed in some of the topics developed in the multiphase space field theory: The use of Schouten brackets in appendix B.5, multiphase space canonical transformations in appendix H.4.1, multiphase space path integrals in section 3.5, BRST and the use of extended multiphase space BRST for systems with secondary constraints in section 4.4, which is applied to the electromagnetic and Yang-Mills fields in section 4.6.4. In addition the BRST-like geometry of the Witten topological sigma model is explored in chapters 5 and 6.

1.1 Background and justification

The basic principle of special relativity is the notion that space and time should be on the same footing, and that the familiar rotation and translation symmetries of space are extended to spacetime as the Poincare group. When we incorporate special relativity, the quantum mechanics of fundamental particles lead us to quantum field theory (QFT) where the fundamental objects are local fields [88]. However the canonical quantization approach to QFT does not embody the principle of spacetime covariance, because it is based on classical Hamiltonian mechanics, where time has a privileged role.

In conventional phase-space Hamiltonian mechanics applied to field theory, a field is treated simply as a Hamiltonian mechanical system with an infinite number of degrees of freedom (the field value at all spatial points), with the spatial structure of the field treated differently from the temporal, and in which the Hamiltonian is a functional of a time slice of the field in space-time. Hamiltonian mechanics thus gives time a special significance, whereas space only appears as a parametrization of degrees of freedom, and so the Hamiltonian formalism does not incorporate what may be (and are usually assumed to be) fundamental aspects of physical reality: spacetime covariance and locality. The particular physical models studied may have these properties, but the Hamiltonian formalism cuts across this and is, in a sense, too broad. The multisymplectic formalism inherently embodies the locality property of the fields on the spacetime background and, in addition, treats time in the same way as the spatial parameters of the field.

It may be that a covariant Hamiltonian mechanics is needed for quantum field theory with general covariance or true background independence [21]. In addition it might be possible to generalize the canonical quantization of classical Hamiltonian field theories in phase space to an analogous 'canonical quantization' in multiphase space [94]. The Feynman functional integration approach to quantum field theory may be re-expressed in multiphase space and a covariant Hamiltonian approach may have a place in the interpretation of QFT, as well as in practical analysis of specific systems. This is suggested in section 3.5.

In this thesis we explain the multiphase space approach - which is a covariant generalization of Hamiltonian mechanics for field theories which is both intrinsically local and covariant. In particular, we then construct the Hamilton-Jacobi theory, the use of the analogue of Poisson brackets, systems with gauge symmetries, and the BRST method for dealing with gauge symmetries, with Yang-Mills theory as an example.

A new formulation of a theory, such as Yang-Mills field theory, can be useful for simplifying the analysis and may capture better certain features such as, in the multiphase-space case, the geometrical properties of locality and relativistic covariance. The multiphase-space formulation of field theory is more restrictive than phase-space mechanics in that it requires the fields to be local. A less general formalism may be more useful for the category of objects it can deal with. For instance, multisymplectic methods are less general, in the sense that it is suited only to local field theories, than symplectic Hamiltonian mechanics, and has the advantage that the multiphase space is a finite dimensional bundle - reflecting the bundle structure of fields - and not infinite dimensional, as is the phase space for a field. Another reason is that the conceptual framework may be more appropriate for that category of objects in that it may lead to fruitful modifications or generalizations.

The multisymplectic formalism is perhaps most conveniently introduced as a generalization of Hamiltonian mechanics and for this it is convenient at this point to be specific and describe the basic features of Hamiltonian (symplectic) mechanics so as to clearly present how multisymplectic mechanics is a generalization appropriate for field theory.

1.1.1 Multiphase space contrasted with phase space

A more abstract and detailed summarization of phase space (symplectic) mechanics is given in chapter 2 as a preparation for the description of analogous structures in multiphase space (multisymplectic) mechanics given in chapter 3.

Phase space

In Hamiltonian mechanics, a point in *phase space*, with local coordinates $(q^i, p_i)_{i=1\ldots N}$, where N is the number of degrees of freedom, specifies the state of a dynamical system. The N q^i's are the configuration variables which describe the configuration or 'position' information of the system, and the N p_i's are the momenta, which contains information about the quantity of 'motion' at a given time. A specific physical system will be characterized by a function $H = H(\bar{q}, \bar{p})$ on phase space, called the *Hamiltonian*. Given a Hamiltonian, this information (the 'state' $(\bar{q}, \bar{p})$) is enough (in the classical description of the world) to determine, in closed systems, the future states of the system: the time evolution of a dynamical system is represented by a path specified by functions of time, $(\bar{q}, \bar{p}) = (q^i(t), p_i(t))_{i=1\ldots N}$, into phase space, where t is the time parameter. The Hamiltonian determines which time-parametrized paths on phase space are allowed evolution trajectories. This is expressed by Hamilton's equations of motion:

$$\frac{\mathrm{d}q^i}{\mathrm{d}t} = \frac{\partial H}{\partial p_i} \quad , \quad \frac{\mathrm{d}p_i}{\mathrm{d}t} = -\frac{\partial H}{\partial q^i}, \tag{1.1}$$

(There are 2N equations as the index i runs from $i = 1\ldots N$.)[1] The phase space can be expressed more geometrically, in a coordinate free manner, as $\mathfrak{T}^*Q$, the dual tangent bundle over the configuration space Q, the manifold with local coordinates $q^i, i = 1\ldots N$. On $\mathfrak{T}^*Q$ can be defined a canonical *symplectic form* ω, in locally adapted coordinates $\omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$, where $\mathrm{d}q^i, \mathrm{d}p_i$ are the coordinate basis 1-forms on $\mathfrak{T}^*Q$. (The summation convention, with identical upper and lower indices being summed over, is in force throughout this thesis. Thus $\mathrm{d}q^i \wedge \mathrm{d}p_i := \sum_{i=1}^N \mathrm{d}q^i \wedge \mathrm{d}p_i$). A fiber bundle can be defined where the phase space is the fibre over a base space $\mathbb{R}$, which represents time. The path $(q^i(t), p_i(t))$, is then a section of the phase space bundle $\mathfrak{T}^*Q \times \mathbb{R}$, and the Hamiltonian function $H = H(\bar{q}, \bar{p}, t)$ is a function on this bundle. Hamilton's equations (1.1), which define the physical trajectories of the system can be written in a coordinate free way as an algebraic equation of differential geometry, $\omega(\bar{v}) = \mathrm{d}H$, where d is the exterior derivative. Thus, at any point on the bundle $\mathfrak{T}^*Q$, the contraction of the velocity vector $\bar{v}$ of a trajectory with the symplectic form is equal to the 1-form defined by the exterior derivative, on the fiber $\mathfrak{T}^*Q$, of the Hamiltonian function H. This phase space structure can be generalized to a notion of symplectic manifold, which is defined in the next

[1] In this thesis the fact that all the index values are assumed in an expression will not usually be explicitly indicated. Thus a point in phase space will be shown as (q^i, p_i) rather than $(q^i, p_i)_{i=1\ldots N}$.

chapter.

Multiphase space

For a field theory on a d-dimensional spacetime B, the multiphase space formalism employs, as the generalization of the symplectic form, a $d+1$-form Ω, called the *multisymplectic form*, on a *covariant multiphase space* M, which is a generalization of the phase space bundle $\mathfrak{T}^*Q \times \mathbb{R}$. Multiphase space M is a bundle over the base space B, with local coordinates $(x^\mu, u^i, p_i^\mu)_{\mu=1\ldots d\ ;\ i=1\ldots N}$, where x^μ are local coordinates of the spacetime manifold B, u^i are the N field values and p_i^μ is the spacetime *multimomentum* of the i^{th} field in the μ direction. When $\mu = t$, the time direction, $p_i^t(x)$ is the ordinary canonical momentum of the i-th field at x, and, for μ a spatial direction, $p_i^\mu(x)$ is the stress at x of the i^{th} field in the μ^{th} direction. Thus multimomenta are 'momenta' with a spacetime index μ, representing the direction in spacetime that the momentum is measured in the field at a given point x in spacetime. In the case of a one dimensional spacetime ($d = 1$), Ω is a 2-form on the covariant multiphase space $\mathsf{M} = \{(t, u^i, p_i^t)\}$. Thus, in the latter case of no spatial dimensions, Ω reduces to the symplectic form ω, and M the ordinary phase space bundle $\mathfrak{T}^*Q \times \mathbb{R}$ of Hamiltonian mechanics.

The equations, which relate the multimomenta p_i^μ to the rate of change of the field configuration, are obtained from the Legendre transformation of the Lagrangian density $\mathcal{L}(u^i, u^i_{,\mu}, x)$, ($u^i_{,\mu}$ stands for $\frac{\partial u^i}{\partial x^\mu}$ at each point $x \in B$ in spacetime when we consider a particular field configuration $u(x)$) which specifies the system: $p_i^\mu = \frac{\partial \mathcal{L}}{\partial u^i_{,\mu}}$. This is a generalization of the way that momenta are obtained from the Lagrangian by the Legendre transformation, $p_i = \frac{\partial L}{\partial u^i_{,t}}$, in phase space mechanics. In a field theory, $p_i^0 = \frac{\partial \mathcal{L}}{\partial u^i_{,0}}$ is the momentum density of the field for $\mu = 0$ (the temporal direction t), and $p_i^\mu = \frac{\partial \mathcal{L}}{\partial u^i_{,\mu}}$ is the stress density for $\mu = 1, 2, 3$ (the spatial directions x, y, z). The analogue to the Hamiltonian $H(u^i, p_i, t)$ and Hamilton's equations of motion in ordinary Hamiltonian mechanics is, in the multiphase space formalism, the DeDonder-Weyl 'Hamiltonian' density $\mathcal{H}(u^i, p_i^\mu, x^\mu)$ and the DeDonder-Weyl equations of motion:

$$\begin{aligned} \partial_\mu u^i(x) &= \frac{\partial \mathcal{H}}{\partial p_i^\mu}(x, u^j(x), p_j^\nu(x)) \\ \partial_\mu p_i^\mu(x) &= -\frac{\partial \mathcal{H}}{\partial u^i}(x, u^j(x), p_j^\nu(x)) \end{aligned} \tag{1.2}$$

These are equivalent (in the regular case) to the usual Euler-Lagrange equations for $\mathcal{L}(u^i, u^i_{,\mu}, x)$ above, (and the DDW 'Hamiltonian' is obtained from $\mathcal{H}(x^\mu, u^i, p_i^\mu) = p_i^\mu \partial_\mu u^i - \mathcal{L}$, similarly to how a Hamiltonian is obtained from a Lagrangian: $H(t, q^i, p_i) = p_i \dot{q}^i - L$).

This multiphase-space structure can be generalized to a notion of multisymplectic manifold, which is defined in chapter 3.

There is a generalization of the coordinate free version, $\omega(\bar{v}) = \mathrm{d}H$, of Hamilton's equations. In the multisymplectic formalism: $\Omega(\bar{\bar{X}}) = \mathrm{d}\mathcal{H}$, where $\bar{\bar{X}}$ is a multivector tangent to a field configuration section in multiphase space [2] [41]. However for the latter, because the covariant multiphase space Hamilton's equations, the DeDonder-Weyl equations of motion, are partial differential equations rather than ordinary differential equations as in Hamilton's equations, there are extra complications which are examined in appendix E, 'Multivector Picture'.

In fact, multimomenta turn up 'naturally' in the study of fields by choosing certain canonical pairs of observables in the classical actions: The electromagnetic and Yang-Mills fields $F_a^{\mu\nu}$ are the (antisymmetrized) multimomenta for the potentials A_ν^a, the Christoffel symbols $\Gamma^\mu_{\lambda\nu}$ are (linear combination of) multimomenta for the metric density $\bar{g}_{\mu\nu} := (-|g|)^{-\frac{1}{2}} g_{\mu\nu}$ in the action for General Relativity. These multimomenta are often employed as convenient objects in the analysis of these dynamical systems even when the authors are not employing a multiphase space formalism. These examples and others are examined in this paper, in the full multiphase space, in sections 3.8.

Many of the structures of classical symplectic mechanics have their analogous multisymplectic formulations: 'Hamiltonian' functions, 'Hamiltonian' equations of motion, Legendre transformation, phase-space Lagrangians, variational principles, constraints, symmetry, reduction, etc., and many papers on the multisymplectic approach aim at these generalizations. However, there are restrictions on the possible form of DeDonderWeyl 'Hamiltonians' - unlike the case for Hamiltonian functions on symplectic manifolds. Also, the use of higher degree forms for observables, such as the multimomentum $d-1$ form $p_i^\mu \mathrm{d}^{d-1}x_\mu$ (see section 1.3 for this notation), as opposed to functions (0-forms) on the phase space in symplectic mechanics, lead to more complicated algebras and difficulties in defining the analogue of the Poisson bracket of functions on the symplectic phase space. A Poisson bracket is desirable, one reason being that canonical quantization involves constructing a representation on a Hilbert space as a Lie algebra of a subalgebra of the Poisson algebra of observables (functions) on a classical phase space. Another reason is that a Poisson bracket maps observables to derivations on the algebra of observables, which can be infinitesimal symmetries or other transformations such as time evolution - which is a key property of phase space and symplectic manifolds generally.

The multisymplectic formalism puts limitations on the type of theories which can be usefully described employing it. Locality and spacetime symmetry are enforced by the tensorial character of observables etc. It places restrictions on the form of DDW Hamiltonians. The theory is far more restrictive than Hamiltonian mechanics, however the restrictions correspond to physical principles such as locality and covariance.

Just as Hamilton's equations of motion are the Euler-Lagrange equations of motion for the extremization of an action which is the time integral of a *phase-space Lagrangian* which contains the Hamiltonian (called the first order formalism), the analogous DeDonder-Weyl equations of motion are the Euler-Lagrange equations of motion for the extremization of an action which is the spacetime integral of a multiphase-space Lagrangian density which contains the DeDonder-Weyl 'Hamiltonian'. (This multiphase space first order formalism is a natural formulation employed in the Palatini formulation of General Relativity and in classical Yang-Mills theory). In this thesis, the first order formalism will frequently be employed to express Hamiltonian dynamics and its multisymplectic generalization where the Euler-Lagrange equations are the Hamilton's or DeDonder-Weyl equations respectively. This will serve several purposes: it clarifies the meaning and has natural generalizations from symplectic to multisymplectic formalism, and it also appears in the construction of the quantum path integral.

1.1.2 Symmetry and gauge theories

An important issue in classical mechanics is symmetry and constraints in dynamical systems. This is important in fundamental physics as many systems explicating physical reality, such as general relativity and the standard model, are best described by gauge theories where the number of physical degrees of freedom is less than the number of degrees of freedom used to describe or define the system initially, or in a simple way. How this is dealt with in multiphase space has to be investigated. The modern technique of BRST, which is important in quantizing gauge fields and in string theory, can be generalized to the multiphase space setting, as is shown in chapter 4, and applied to the Yang-Mills field in that chapter. This may be of use in quantum field theory (QFT) because multiphase space can be very naturally employed in the functional integrals employed in QFT. Algebraic methods like BRST which employ the natural Poisson structure of phase space have several advantages over analytical approaches: the inherent power of algebra, compatibility with the structure of canonical quantization, the implementation of the Lie algebra of symmetry groups as Poisson structures, the power of algebraic geometry, the use of homological methods. The BRST methods work in both classical and quantum mechanics and there is a motivation in exploring whether multisymplectic methods can be applied here.

1.1.3 Quantum field theory

In canonical quantization, the starting point is Hamiltonian mechanics and, for a classical field, canonical quantization again singles out time and put the spatial structure of the field on a different footing. It may be that multisymplectic dynamics could be the starting point to a form of quantization for fields analogous to canonical quantization where covariance and locality enters in an essential way. Kanatchikov [48] [95] [94] has attempted to use multisymplectic ideas to generalize quantum mechanics. In quantum field theory, the multimomenta in the multiphase space functional integral of the complex exponential of the classical action expressed as the spacetime integral of a multiphase space Lagrangian density can be integrated out. This is the same way that momenta are integrated out from the phase space functional integral, to give the usual functional integral of field configurations in QFT. It could therefore of interest to see if the multiphase space functional integral can be constructed from some kind of spacetime operator formulation of QFT, just as, in introductory QFT texts, the phase space path integral is usually constructed by considering the product of a temporal sequence of infinitesimal evolution operators in quantum mechanics (section 3.5).

1.1.4 Topological sigma model

It was pointed out by S. Hrabak [83] that, in the Witten paper on the topological sigma model [30], there are auxiliary fields which are similar to multimomenta, and the Lagrangian Witten constructs is similar to a multiphase space action Lagrangian. As Witten points out in [30], his construction is similar to BRST. The seeming multiphase space BRST formulation in Witten's paper was done without employing any multiphase space technology, and this thesis attempts to show partway how the Witten model could be constructed as a multiphase space BRST extension of a classical system with gauge symmetries (in this model, J-holomorphic variations of the embedding of a Riemann surface in a almost Hermitian maniforld), and gives some conjectures on how the construction could be completed. The geometrical background of the required multiphase space BRST formulation is in chapter 5 and applied to the topological sigma model in chapter 6.

1.1.5 Notes

In most of the discussion and examples a flat Lorentzian or Riemannian spacetime is assumed. However, as the multiphase space formalism is covariant, it is readily adaptable to general coordinates on curved spacetimes. Rather than developing the formalism to seek generality we present it in a 'physicist's' viewpoint where restricted practical applications such as Yang-Mills models are in view. Much of the mathematical description of the fields and the underlying geometry is presented employing coordinates, which should be assumed to be local coordinates on coordinate patches on a manifold or bundle. A presentation which employs more abstract geometrical notation is avoided except when there are notions where the geometry involved brings clarity as when it can be seen to be a generalization of the geometrical view of classical mechanics. For this reason, chapter 2 is included, so as to cast the well known Hamiltonian mechanics and field theory in a geometrical form, where the parallels with multisymplectic mechanics in chapter 3 will be more readily seen. The use of the term 'multisymplectic' in the expression 'multisymplectic field theory' does not imply that all multisymplectic manifolds can work as a multiphase space in any particular aspect of multisymplectic field theory, any more than we can expect all parts of classical mechanics to be well defined on all symplectic manifolds. In the latter one generally has in view the tangent bundles of the manifolds of the values of the configuration parameters of classical mechanical systems. Similarly, in multisymplectic mechanics, one has in view the the dual jet bundles of the jet bundles (field values and their spacetime derivatives) over spacetime of the fields in a field theory. We also concerned only with first order Lagrangians, i.e. which only depending on first order time derivatives (and for Lagrangian densities in field theory only depending on first order space and time derivatives). We also usually assume that maps and manifolds are smooth.

1.1.6 Conventions

The conventions in force in this thesis, as well as some differential-geometric identities, are listed at the end of this chapter in section 1.3. Appendix A contains information about different types of almost complex manifolds useful for the Witten model in chapters 5 and 6.

1.1.7 Brief history of multiphase space methods

Volterra [96] [97] is considered to have started the multiphase-space project in the context of the calculus of variations where he generalized to several variables the Hamilton equations for variational problems which were then called the Hamilton-Volterra equations. These are now known as the DeDonder-Weyl equations. The approach through the calculus of variations over several variables was followed by Caratheodory [19] and is now called Caratheodory theory. A variation of this was developed by Weyl [42] and is called DeDonder-Weyl theory. Cartan [27] and DeDonder [89] studied Hamilton's equations starting from the theory of invariant integrals. Lepage [91] who made these part of a larger theory. A geometrical approach to the latter was taken by Dedecker [74]. Symplectic and geometical developments were made by Garcia [76], Goldschmidt and Sternberg [82], Krupka [22]. The geometric multisymplectic ideas in their current form were started by Kijowski [57] and Goldschmidt and Sternberg [82], who first defined a multisymplectic manifold. Further early developments were by Helein [32], Kijowski and Tulczyjew [100] [56], Kijowski and Szczyrba [63], Martin [34], Marsden and Weinstein [6], Sardanashvily [35], Kanatchikov [93]. Multisymplectic methods for higher order field theories developed by Gotay [28] and also Kouranbaeva and Shkoller [86]. The problem of defining Poisson brackets on observables on multiphase space is examined in papers by Szczyrba and

Kijowski [63], Kanatchikov [47], and is related to classifying observables: Helein [32], Kouneiher [55], Cantrijn [31]. Consructing a generalization of BRST to multiphase space was done by Hrabak [83] [84]. There are several papers of application to water waves: Bridges [51], Marsden and Shkoller, [87]. There is a paper of application to continuum mechanics: Marsden, Pekarsky, Shkoller, and West [69]. On the variational bicomplex, which is an algebraic approach to Lagrangian field theory, there are papers by Vinogradov [10], [11], Tulczyjew [101], Bridges [52]. There are several papers on application to General Relativity: Fradkin [33], Vey [24], Ashtekar, Bombelli, and Reula [72], Anco and Tung [79], Rovelli [21]. There are some modern reviews of the topic: Gotay [28], Helein [54].

1.2 Summary of the thesis

Chapter 2 summarises classical mechanics in a form from which the generalization to the multisymplectic formalism will be more apparent, expressing it to a large extent in the formalism of differential geometry on symplectic manifolds. We start with Hamiltonian mechanics on phase space and on extended phase space, where the latter includes an extra canonical pair of coordinates: time and energy. Lagrangian mechanics is presented using the differential geometry of forms. In particular, the action is shown to be the integral of the canonical one-form along the trajectory in extended phase space (this is the first order formalism with a phase-space Lagrangian expressed using differential geometry on symplectic manifolds). This continues by summarizing how symmetries and constraints on the phase space are dealt with. In particular the Marsden-Weinstein reduction [6] to a reduced phase space, where non-physical degrees of freedom are eliminated, is described. Classical mechanics is portrayed in many texts including the classic texts by Arnold [45], Goldstein [39], Landau and Liftshitz [29]. Use has been made of the expositions of symplectic geometry in the books by McDuff [23], Fomenko [3], and Cannas da Silva [25].

Chapter 3 develops the corresponding and analogous formulation of classical field theory in terms of the differential geometry on multisymplectic manifolds [36]. This is described in the same way that classical mechanics was immediately above, with analogous results, because the DeDonder-Weyl Hamiltonian function on multiphase space in classical field theory is a generalization of the Hamiltonian in classical mechanics and plays a similar role. In particular, the action of a field is shown to be the integral of the canonical d-form over the corresponding d-dimensional section in extended multiphase space. Use has been made of the expositions of the multisymplectic approach in the long papers by Paufler [40] and Gotay [28].

Section 3.5 is a brief description of an application of the multiphase space action in the functional integral in quantum field theory.

The theory of constraints and symmetry in multisymplectic mechanics is explained and how Marsden-Weinstein reduction [6] applies in multisymplectic geometry.

The role of brackets is explained and generalizations chosen on multiphase space which could serve in field theory in an analogous way to brackets on phase space as in Kanatchikov [47]. These brackets are used in the multiphase space BRST developed in chapter 4.

The electromagnetic field, Yang-Mills theory, the Bosonic string, and General Relativity are given as examples of the multiphase space formalism. Yang-Mills in chapter 3, the others in the appendix.

Chapter 4 explains the classical BRST construction. Intoductory texts consulted for this chapter were: Figueroa -O'Farill [53]; van Holten [98]; Henneaux, Maas, Fuster [4]; Stasheff [62]; Woit [78]; Henneaux, Teitelboim [20]. The electromagnetic field, based on Nemeschansky [71], is shown as an example. Afterwards is developed the multiphase space analogue. The use of the Hamiltonian as a constraint is presented for dealing with systems with secondary constraints. This is done first in phase space then in multiphase space, the latter using the 'hybrid technique' where integrals over spatial sections are employed. This is then applied to Yang-Mills fields. Use was made of Rogers [8].

Chapter 5 applies multiphase space BRST to a Riemannian manifold target space (Rogers [9]). This structure is applied in the chapter 6.

Chapter 6 was intended to show that the Witten topological sigma model [30] could be recreated as the multiphase space BRST modification of a bosonic model. This has not been achieved so far, so the geometrical aspects which have been studied are presented.

Following chapter 6 are the appendices:

Appendix A: Lists the identities of the almost complex structure J which are relevant for the almost hermitian manifolds in the Witten theory of chapter 7. This was developed from the papers by Gray [37] [5].

Appendix B: Deals with brackets. The basic definition and properties of the particular multi-bracket in general use throughout this paper which we call the 'multi-Poisson bracket'. A brief description of various Lie brackets in differential geometry is given, including the Schouten bracket. The paper by Michor and Dubois-Violette [75] gives basic relationship between various kinds of brackets.

Appendix C: In this appendix, the conventional phase space presentation of the electromagnetic field is laid out to contrast with the multiphase space version which is in appendix D: 'Other multiphase space examples'. In particular how gauge symmetry is treated. This is a detailed exposition the electomagnetic field following Nemeschansky [71] and Barut [12], which leads on to the phase space BRST example in section 4.3.1.

Appendix D: In this appendix are developed, to varying degree, examples of fields expressed in multiphase space: Scalar fields with global symmetry. The electromagnetic field where an example of multiphase space Marsden-Weinstein symplectic reduction is presented in detail. The Bosonic string and General Relativity are also presented with the DeDonder-Weyl Hamiltonian and the multiphase-space energy-momentum tensor are derived. (Note: the Yang-Mills field is presented in detail in Chapter 3 and not in appendix D.)

Appendix E: The notion of a multivector field to characterize the tangent space of field solution sections of multiphase space is examined as generalization of the vector fields in phase space. One of the significant differences between symplectic and multisymplectic mechanics is that, while an infinitesimal symmetry variation is represented as a vector field in both, an infinitesimal time evolution is a vector field only in phase space: in multisymplectic mechanics, the spacetime evolution of a field is a section, over spacetime, of multiphase space, which can be represented by a multivector fields on the section - and is not a vector field. Another way of viewing this is that phase space is foliated into trajectories by the Hamiltonian, but sections satisfying the equations of motion can intersect and do not foliate multiphase space. Thus the foliation property of phase space to parametrize solutions does not generalize to multiphase space. Therefore it is of interest to find ways of characterizing solutions. The Hamilton-Jacobi theory in appendix H is also interesting for this reason. This was developed by Paufler, Romer and Forger in [38] and [41].

Appendix F: The theory of canonical transformations in phase space and the use of generating functions explained in the text by Goldstein [39]. This is generalized to multiphase space in appendix H.4.1.

Appendix G: The phase space Hamilton-Jacobi equations are explained and shown to be the closure condition of certain forms on the extended configuration space [39], this is generalized to multisymplectic Hamilton-Jacobi theory in appendix H.

Appendix H: In the multisymplectic Hamilton-Jacobi theory, the generalized Hamilton-Jacobi equations are expressed as the closure of certain forms on the field configuration space. This is given in the papers by Kastrup [43] [44] and Marrero, De Diego, De Leon [66]. In addition the generalization of generators for canonical transformation is examined and it is shown that there is only one type as opposed to four types in symplectic mechanics. In this appendix, original work on the method of generating functions for canonical transformations in multiphse space is shown.

Appendix I is the bibliography

1.3 Notation, conventions, terminology

The expressions 'phase-space mechanics', 'Hamiltonian mechanics', 'symplectic mechanics', all mean the same thing in this thesis.

The expressions 'multiphase-space mechanics', 'multiphase-space field theory', 'multisymplectic mechanics', 'multisymplectic field theory', all mean the same thing in this thesis.

We spell the phrase 'multiphase-space' with a hyphen when 'multiphase-space' is an adjectival phrase and 'multiphase space' when this phrase serves as a noun. Similarly for the phrase 'phase-space'.

We reserve the use of upper case 'Hamiltonian' for the function on phase space which defines the time evolution of a dynamical system. We use lower case 'hamiltonian' for a function on phase space which generates a vector field or a symmetry. Similarly in the multiphase space setting.

'Path' refers to any curve in configuration space, phase space or any section of the bundle of the field configuration, jet bundle, or multiphase space over spacetime, etc, depending on the context. Note that the word 'path' is also use for the time evolution of a field. 'Trajectory' refers to any path which satisfies the equations of motion, i.e. the Euler-Lagrange equations, Hamiltons equations, stationarity of the action, etc, depending on the context. Similarly, for fields, 'configuration' and 'path' is also used to mean any field evolution, whereas 'trajectory' is a field which satisfies the field equations or the variational principle. The weak equality $\approx$ is used to indicate that the path or field configuration satisfies the equations of motion or the variational principle. It is also sometimes used to indicate a constraint is in force.

The Einstein summation convention, whereby any pair of indices, one upper, one lower, which are the same symbol, represent the sum of that term, for each value over the range of that symbol. E.g.: $\delta^\mu_\mu = \delta^0_0 + \delta^1_1 + \delta^2_2 + \delta^3_3 = 1 + 1 + 1 + 1 = 4$, for a 4-dimensional Minkovski metric. Where the summation convention is suspended, it will be indicated.

The number of dimensions of spacetime is indicated by d. The number of fields is usually N. K is used for the number of symmetries, the dimension of the Lie algebra of the symmetry group.

The notation for a function f on spaces with local coordinates $(x^0, x^1, x^2, \ldots, x^{d-1})$ can be any of the following: $f(x) = f(x^\mu) = f(x^0, x^1, x^2, \ldots, x^{d-1})$. A point in this space can be any of the following: $(x) = (x^\mu) = (x^0, x^1, x^2, \ldots, x^{d-1})$. The entire space or a coordinate patch (depending on the context) can be any of the following: $M^d = \{(x)\} = \{(x^\mu)\} = \{(x^0, x^1, x^2, \ldots, x^{d-1})\}$. Similarly an object with indices such as p_i^μ may mean the object with particular but arbitrary values (μ and i in this example) *or* may mean the object with all values ($\mu = 0, \ldots, d-1$ and $i = 1 \ldots N$ in this example), depending on the context. For example a Lagrangian density may be written in alternative ways $\mathcal{L} = \mathcal{L}(x, u, p) = \mathcal{L}(x^\mu, u^i, p_i^\mu)$.

Differential geometry

The conventions below are for even parity coordinates. For odd parity (grassmann) coordinates, there is an extra minus 1 factor when commuting two odd factors.

$$\mathrm{d}^d x := \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \mathrm{d}x^{d-2} \wedge \mathrm{d}x^{d-1} = (-1)^{d(d-1)/2} \mathrm{d}x^{d-1} \wedge \mathrm{d}x^{d-2} \wedge \ldots \wedge \mathrm{d}x^2 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^0$$

$${}^d\partial := \tfrac{\partial}{\partial x^{d-1}} \wedge \tfrac{\partial}{\partial x^{d-2}} \wedge \ldots \wedge \tfrac{\partial}{\partial x^2} \wedge \tfrac{\partial}{\partial x^1} \wedge \tfrac{\partial}{\partial x^0} = (-1)^{d(d-1)/2} \tfrac{\partial}{\partial x^0} \wedge \tfrac{\partial}{\partial x^1} \wedge \tfrac{\partial}{\partial x^2} \wedge \ldots \wedge \tfrac{\partial}{\partial x^{d-2}} \wedge \tfrac{\partial}{\partial x^{d-1}}$$

$$\partial^\alpha := (-1)^\alpha \tfrac{\partial}{\partial x^{d-1}} \wedge \tfrac{\partial}{\partial x^{d-2}} \wedge \ldots \wedge \widehat{\tfrac{\partial}{\partial x^\alpha}} \wedge \ldots \wedge \tfrac{\partial}{\partial x^2} \wedge \tfrac{\partial}{\partial x^1} \wedge \tfrac{\partial}{\partial x^0}$$

$$= (-1)^{d-(\alpha+1)} \tfrac{\partial}{\partial x^0} \wedge \tfrac{\partial}{\partial x^1} \wedge \tfrac{\partial}{\partial x^2} \wedge \ldots \wedge \widehat{\tfrac{\partial}{\partial x^\alpha}} \wedge \ldots \wedge \tfrac{\partial}{\partial x^{d-2}} \wedge \tfrac{\partial}{\partial x^{d-1}} = {}^d\partial \llcorner \mathrm{d}x^\alpha = (-1)^d \, \mathrm{d}x^\alpha \lrcorner {}^d\partial$$

This last is the same as the definition of ${}^d\partial$ except one of the factors, $\widehat{\frac{\partial}{\partial x^\alpha}}$, in the wedge product is omitted. The hat above a factor in the wedge product indicates that the factor is not present.

Writing coordinate basis vectors more compactly: $\partial_0 := \frac{\partial}{\partial x^0}$, $\partial_\alpha := \frac{\partial}{\partial x^\alpha}$, etc.,

$${}^d\partial = (-1)^{(d-1-\alpha)} \partial_\alpha \wedge \partial^\alpha = (-1)^\alpha \partial^\alpha \wedge \partial_\alpha \quad \text{(no summation over } \alpha \text{)}$$

$${}^d\partial \ \delta^\alpha_\beta = (-1)^{(d-1-\alpha)} \partial_\beta \wedge \partial^\alpha = (-1)^\alpha \partial^\alpha \wedge \partial_\beta$$

The convention for contraction is that contraction is positive if the basis vector is on the left of the basis 1-form, negative if the basis vector is on the right of the basis 1-form, and they are adjacent. A minus 1 factor is picked up when switching basis vectors or forms.

The left acting exterior derivative $\overleftarrow{\mathrm{d}}$, has the following properties: $f\overleftarrow{\mathrm{d}} = -\mathrm{d}f$ on a scalar f, and $\omega^{(p)}\overleftarrow{\mathrm{d}} = -(-1)^p r\mathrm{d}\omega^{(p)}$. It obeys the form graded Liebnitz rule.

When contracting into the forms, all the vectors have to be absorbed for a non-zero result, and the result is a form: $\partial_\alpha \lrcorner\, \mathrm{d}x^\beta = -\mathrm{d}x^\beta \llcorner\, \partial_\alpha = \delta_\alpha^\beta$

$\partial_2 \lrcorner\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \llcorner\, \partial_2 = -\mathrm{d}x^1$

$\partial_2 \wedge \partial_1 \lrcorner\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \llcorner\, \partial_2 \wedge \partial_1 = 1$

$\partial_2 \wedge \partial_1 \lrcorner\, \mathrm{d}x^2 = \mathrm{d}x^2 \llcorner\, \partial_2 \wedge \partial_1 = 0$

$1 \lrcorner\, \mathrm{d}x^\alpha = \mathrm{d}x^\alpha$

When contracting into the vectors, all the forms have to be absorbed for a non-zero result, and the result is a multivector: we have $\partial_\alpha \llcorner\, \mathrm{d}x^\beta = -\mathrm{d}x^\beta \lrcorner\, \partial_\alpha = \delta_\alpha^\beta$

$\partial_2 \llcorner\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \lrcorner\, \partial_2 = 0$

$\partial_2 \wedge \partial_1 \llcorner\, \mathrm{d}x^1 \wedge \mathrm{d}x^2 = \mathrm{d}x^1 \wedge \mathrm{d}x^2 \lrcorner\, \partial_2 \wedge \partial_1 = 1$

$\partial_2 \wedge \partial_1 \llcorner\, \mathrm{d}x^2 = \mathrm{d}x^2 \lrcorner\, \partial_2 \wedge \partial_1 = -\partial_1$

$1 \llcorner\, \mathrm{d}x^\alpha = 0$

$1 \llcorner\, \partial_\alpha = 0$

$1 \lrcorner\, \partial_\alpha = -\partial_\alpha \llcorner\, 1 = \partial_\alpha$

In general

$v \llcorner\, \omega = (-1)^{|v||\omega|} \omega \lrcorner\, v$

Note that the -1 factor above arises from commuting the vector and form, not from reversing the contraction operator.

$\mathrm{d}x_\alpha := (-1)^\alpha\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots\ldots \wedge \widehat{\mathrm{d}x^\alpha} \wedge \ldots \wedge \mathrm{d}x^{d-2} \wedge \mathrm{d}x^{d-1} = \partial_\alpha \lrcorner\, \mathrm{d}^d x = (-1)^d\, \mathrm{d}^d x \llcorner\, \partial_\alpha$

With the above definitions,

$\mathrm{d}x^\alpha \wedge \mathrm{d}x_\beta = \mathrm{d}x^\alpha \wedge (\partial_\beta \lrcorner\, \mathrm{d}^d x) = \delta_\beta^\alpha\ \mathrm{d}^d x = (-1)^{d-1}\, \mathrm{d}x_\beta \wedge \mathrm{d}x^\alpha$

$({}^d\partial \llcorner\, \mathrm{d}x^\alpha) \lrcorner\, (\partial_\beta \lrcorner\, \mathrm{d}^d x) = \partial^\alpha \lrcorner\, \mathrm{d}x_\beta = (-1)^{d-1}(\partial_\beta \lrcorner\, \mathrm{d}^d x) \llcorner\, ({}^d\partial \llcorner\, \mathrm{d}x^\alpha) = (-1)^{d-1}\mathrm{d}x_\beta \llcorner\, \partial^\alpha = \delta_\beta^\alpha 1$

$$ {}^d\partial \lrcorner \mathrm{d}^d x = 1 = (-1)^d \ \mathrm{d}^d x \llcorner {}^d\partial $$

$$ \partial^\alpha \wedge \partial_\beta = \delta^\alpha_\beta \ {}^d\partial = (-1)^d \partial_\beta \wedge \partial^\alpha $$

$$ \partial^\alpha \lrcorner \mathrm{d}^d x = (-1)^{d-1-\alpha} \, \mathrm{d}x^\alpha = \mathrm{d}^d x \llcorner \partial^\alpha $$

$$ (\ {}^d\partial \llcorner \mathrm{d}x_\beta) \lrcorner (\partial^\alpha \lrcorner \mathrm{d}^d x) = ((-1)^{(d-1-\alpha)} \ {}^d\partial \llcorner \mathrm{d}x_\beta) \lrcorner ((-1)^{d-1-\alpha} \, \partial^\alpha \lrcorner \mathrm{d}^d x) = \partial_\beta \lrcorner \mathrm{d}x^\alpha = \delta^\alpha_\beta \ 1 $$

$$ \partial_\alpha = (-1)^\alpha \ \mathrm{d}x_\alpha \lrcorner {}^d\partial = (-1)^{(d-1-\alpha)} \ {}^d\partial \llcorner \mathrm{d}x_\alpha $$

Note that the conventions are chosen so that vectors and multivectors on the left contracted to or from forms on the right for some basic expressions give positive results:

$$ \partial_\alpha \lrcorner \mathrm{d}x^\beta \ = \ -\mathrm{d}x^\beta \llcorner \partial_\alpha \ = \ \delta^\beta_\alpha \ \text{ and } \ \partial_\alpha \llcorner \mathrm{d}x^\beta \ = \ -\mathrm{d}x^\beta \lrcorner \partial_\alpha \ = \ \delta^\beta_\alpha $$

$$ \mathrm{d}x_\alpha \ := \ \partial_\alpha \lrcorner \mathrm{d}^d x \ = \ (-1)^d \, \mathrm{d}^d x \llcorner \partial_\alpha \ \text{ and } \ \partial^\alpha \ := \ {}^d\partial \llcorner \mathrm{d}x^\alpha \ = \ (-1)^d \, \mathrm{d}x^\alpha \lrcorner {}^d\partial \ \text{ and} $$

$$ \partial^\alpha \lrcorner \mathrm{d}x_\alpha \ = \ (-1)^{d-1} \, \mathrm{d}x_\alpha \llcorner \partial^\alpha = \ 1 $$

and $ {}^d\partial \lrcorner \mathrm{d}^d x = \ (-1)^d \ \mathrm{d}^d x \llcorner {}^d\partial \ = \ 1 $

with some exceptions:

$$ \partial^\alpha \lrcorner \mathrm{d}^d x = (-1)^{d-1-\alpha} \, \mathrm{d}x^\alpha = \mathrm{d}^d x \llcorner \partial^\alpha \ \text{ and } \ \partial_\alpha = (-1)^\alpha \ \mathrm{d}x_\alpha \lrcorner {}^d\partial = (-1)^{(d-1-\alpha)} \ {}^d\partial \llcorner \mathrm{d}x_\alpha $$

For below we require $\alpha < \beta$:

$$ \mathrm{d}x_{\alpha\beta} := \ \partial_\alpha \lrcorner (\partial_\beta \lrcorner \mathrm{d}^d x) = \ -\partial_\beta \lrcorner (\partial_\alpha \lrcorner \mathrm{d}^d x) = \ \mathrm{d}^d x \llcorner \partial_\alpha \llcorner \partial_\beta = \ -\mathrm{d}^d x \llcorner \partial_\beta \llcorner \partial_\alpha $$

$$ = (-1)^{\alpha-\beta} \, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \widehat{\mathrm{d}x^\alpha} \wedge \ldots\ldots \wedge \widehat{\mathrm{d}x^\beta} \wedge \ldots \wedge \mathrm{d}x^{d-2} \wedge \mathrm{d}x^{d-1} $$

With these definitions

$$ \mathrm{d}x_{\alpha\beta} \ = \ -\mathrm{d}x_{\beta\alpha} $$

$$ \mathrm{d}x^\gamma \wedge \mathrm{d}x_{\alpha\beta} = \mathrm{d}x^\gamma \wedge \ \partial_\alpha \lrcorner (\partial_\beta \lrcorner \mathrm{d}^d x) = \delta^\gamma_\alpha \mathrm{d}x_\beta - \delta^\gamma_\beta \mathrm{d}x_\alpha \ = (-1)^{d-2} \, \mathrm{d}x_{\alpha\beta} \wedge \mathrm{d}x^\gamma $$

$$ \mathrm{d}x^\kappa \wedge \mathrm{d}x^\gamma \wedge \mathrm{d}x_{\alpha\beta} = \mathrm{d}x^\kappa \wedge \mathrm{d}x^\gamma \wedge \ \partial_\alpha \lrcorner (\partial_\beta \lrcorner \mathrm{d}^d x) = \ (\ \delta^\gamma_\alpha \delta^\kappa_\beta - \delta^\gamma_\beta \delta^\kappa_\alpha \,) \ \mathrm{d}x^d = \mathrm{d}x_{\alpha\beta} \wedge \mathrm{d}x^\kappa \wedge \mathrm{d}x^\gamma $$

For below we require $\alpha < \beta < \gamma$:

$$ \mathrm{d}x_{\alpha\beta\gamma} := \ \partial_\alpha \lrcorner (\partial_\beta \lrcorner (\partial_\gamma \lrcorner \mathrm{d}^d x)) = \ (-1)^d \mathrm{d}^d x \llcorner \partial_\alpha \llcorner \partial_\beta \llcorner \partial_\gamma $$

$$= (-1)^{\alpha+\beta+\gamma}\, \mathrm{d}x^0 \wedge \mathrm{d}x^1 \wedge \mathrm{d}x^2 \wedge \ldots \wedge \widehat{\mathrm{d}x^\alpha} \wedge \ldots\ldots \wedge \widehat{\mathrm{d}x^\beta} \wedge \ldots \wedge \widehat{\mathrm{d}x^\gamma} \wedge \ldots \wedge \mathrm{d}x^{d-2} \wedge \mathrm{d}x^{d-1}$$

With these definitions

$$\mathrm{d}x_{\alpha\beta\gamma} \;=\; -\mathrm{d}x_{\beta\alpha\gamma} \;=\; -\mathrm{d}x_{\gamma\beta\alpha} = \mathrm{d}x_{\gamma\alpha\beta}$$

$$\mathrm{d}x^\delta \wedge \mathrm{d}x_{\alpha\beta\gamma} = \mathrm{d}x^\delta \wedge (\partial_\alpha \lrcorner\, \partial_\beta \lrcorner\, \partial_\gamma \lrcorner\, \mathrm{d}^d x) = \delta^\delta_\alpha \mathrm{d}x_{\beta\gamma} + \delta^\delta_\beta \mathrm{d}x_{\gamma\alpha} + \delta^\delta_\gamma \mathrm{d}x_{\alpha\beta} \;= (-1)^{d-3}\, \mathrm{d}x_{\alpha\beta\gamma} \wedge \mathrm{d}x^\delta$$

$$\mathrm{d}x^\kappa \wedge \mathrm{d}x^\delta \wedge \mathrm{d}x_{\alpha\beta\gamma} = \mathrm{d}x^\kappa \wedge \mathrm{d}x^\delta \wedge\, \partial_\alpha \lrcorner\, (\,\partial_\beta \lrcorner\, \partial_\gamma \lrcorner\, \mathrm{d}^d x) = \mathrm{d}x^\kappa \wedge (\delta^\delta_\alpha \mathrm{d}x_{\beta\gamma} + \delta^\delta_\beta \mathrm{d}x_{\gamma\alpha} + \delta^\delta_\gamma \mathrm{d}x_{\alpha\beta}\,)$$

$$= \delta^\delta_\alpha(\delta^\kappa_\beta \mathrm{d}x_\gamma - \delta^\kappa_\gamma \mathrm{d}x_\beta\,) + \delta^\delta_\beta(\delta^\kappa_\gamma \mathrm{d}x_\alpha - \delta^\kappa_\alpha \mathrm{d}x_\gamma\,) + \delta^\delta_\gamma(\delta^\kappa_\alpha \mathrm{d}x_\beta - \delta^\kappa_\beta \mathrm{d}x_\alpha\,)$$

$$= (\delta^\delta_\beta \delta^\kappa_\gamma - \delta^\delta_\gamma \delta^\kappa_\beta)\mathrm{d}x_\alpha + (\delta^\delta_\gamma \delta^\kappa_\alpha - \delta^\delta_\alpha \delta^\kappa_\gamma)\mathrm{d}x_\beta + (\delta^\delta_\alpha \delta^\kappa_\beta - \delta^\delta_\beta \delta^\kappa_\alpha)\mathrm{d}x_\gamma$$

Metric tensor

The covariant derivative of the metric tensor $g := (g_{\mu\nu})$ is zero:

$$\nabla g = 0 \quad \text{therefore} \quad \partial_\mu g_{\nu\kappa} = \Gamma^\lambda_{\mu\nu} g_{\lambda\kappa} + \Gamma^\lambda_{\mu\kappa} g_{\nu\lambda} = 2\Gamma_{\mu(\nu\kappa)} \quad \partial_\mu g^{\nu\kappa} = -2\Gamma^{(\nu\kappa)}_\mu$$

Variation of the determinant $|g| := \det(g_{\mu\nu})$ of the metric tensor:

$$\delta|g| = |g| g^{\mu\nu} \delta g_{\mu\nu} = -|g| g_{\mu\nu} \delta g^{\mu\nu}$$

Covariant derivative and Riemann tensor:

Covariant derivative of a vector v :$(\nabla v)^i_j = \partial_j v^i + \Gamma_{jl}{}^i v^l$

Commutator of covariant derivative of a vector v: $[\nabla_j, \nabla_k]^i{}_l v^l = R_{jk}{}^i{}_l\; v^l$

$$R_{jk}{}^i{}_l \;=\; \partial_j \Gamma_{kl}{}^i \;-\; \partial_k \Gamma_{jl}{}^i \;+\; \Gamma_{js}{}^i\, \Gamma_{kl}{}^s -\; \Gamma_{ks}{}^i\, \Gamma_{jl}{}^s$$

$$R_{jk}{}^i{}_l \;= R^i{}_{ljk} \;= -R_{kj}{}^i{}_l \;= -R^i{}_{lkj} \;=\; -R_{jkl}{}^i \;= -R_l{}^i{}_{jk} \;=\; R_{kjl}{}^i \;= R_l{}^i{}_{kj}$$

Metric density, which is the metric tensor multiplied by the square root of the determinant of the metric, used in appendix D.4:

$$\bar{g}^{\mu\nu} := (-|g|)^{\frac{1}{2}} g^{\mu\nu}$$

$$\bar{g}_{\mu\nu} := (-|g|)^{-\frac{1}{2}} g_{\mu\nu}$$

$\delta\bar{g}^{\mu\nu} = (-|g|)^{\frac{1}{2}}(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu})\delta g^{\alpha\beta}$

$\delta g^{\mu\nu} = (-|g|)^{-\frac{1}{2}}(\delta^{\mu}_{\alpha}\delta^{\nu}_{\beta} - \frac{1}{2}g_{\alpha\beta}g^{\mu\nu})\delta\bar{g}^{\mu\nu}$

Chapter 2

Classical mechanics and field theory

In this chapter classical mechanics and field theory will be presented and discussed in the formalism of the differential geometry of symplectic manifolds. From this the generalization to multisympectic field theory in chapter 3 will be apparent and straightforward.

We start with the well known Hamilton's equations of motion on phase space and express them in geometrical (coordinate free) notation in the differential geometry of forms on a symplectic manifold. This brings out the mathematical structure that will be generalized in the next chapter on multisymplectic mechanics. The subject of symplectic geometry is only touched upon in this thesis for the purposes of describing mechanics and little of the area of symplectic geometry [3] is touched upon. Lagrangian mechanics is also presented in the notation of differential geometry with, in particular, the phase-space action, which generalizes to the multiphase-space action explained in the parallel section in the chapter following. The chapter ends with the examination of systems with symmetry as its generalization to the multiphase-space setting is the major theme of this thesis. Canonical transformations and Hamilton-Jacobi theory are described in the appendix as well as the example of the electromagnetic field which is taken up as an example in the chapter on BRST.

2.1 Hamiltonian mechanics

In a classical mechanical system specified by a Hamiltonian function $H(q^i, p_i, t)$ in local coordinates, Hamilton's equations of motion is the following set of $2N$ first order ordinary differential equations, which are conditions which hold at each instant t of a trajectory[1] in phase space:

$$\dot{q}^i \approx \left(\frac{\partial H}{\partial p_i}\right) \quad , \quad \dot{p}_i \approx -\left(\frac{\partial H}{\partial q^i}\right), \tag{2.1}$$

where the $(q^i)_{i=1\ldots N}$ are N configuration dynamical variables which are local coordinates of the configuration space Q, and $(p_i)_{i=1\ldots N}$ are the momentum dynamical variables canonically dual to the q^i. $(q^i, p_i)_{i=1\ldots N}$ are local adapted coordinates on the classical phase space of the system, $\mathfrak{T}^*Q$, which is the bundle dual to the tangent bundle $\mathfrak{T}Q$ of the configuration space Q. $H = H(q^i, p_i, t)$ is a function on $\mathfrak{T}^*Q \times \mathbb{R}$, called the Hamiltonian function and needs to be specified for each dynamical system. The left hand sides of the equations above, $(\dot{q}^i, \dot{p}_i)$, are the time rates of change of the dynamical variables (q^i, p_i) and is meant to indicate the instantaneous rate of change at a point on the path . The weak equality symbol $\approx$ is used to show that the equality only holds for particular paths $(q^i, p_i)(t)$ - those that satisfy the equations of motion (which we will call trajectories) and are usually the physical trajectories of interest. The points of phase space represent the states of the (classical) dynamical system, and functions (such as for example $H(q^i, p_i, t)$) on phase space are called observables because they are variables whose value depends on the state. Since the partial derivatives on the right hand side of (2.1) are functions on phase space, $(\dot{q}^i, \dot{p}_i)$ (more precisely $\dot{q}^i\left(\frac{\partial}{\partial q^i}\right)+\dot{p}_i\left(\frac{\partial}{\partial p_i}\right)$), is a vector field defined by (2.1) on all of phase space. (Note that, in this thesis, summation over a repeated upper and lower index in a product, such as the i's here, is implied.) The integral parametrized curves, which can always be constructed from a vector field, are the trajectories in phase space parametrized by time. Each point in phase space is on a unique trajectory, so every point in phase space is a state which is the starting point of a unique trajectory. This is a general property of a system of first order ordinary differential equations such as (2.1). Thus name 'state' is appropriate - because a specified state here specifies the subsequent (and preceding trajectory or evolution of the system).

If the first set of equations can be solved for the momenta p^i, we can substitute for $p_i = p_i(q^j, \dot{q}^j)$ in the second set of equations and so obtain the configuration space equations of motion which will, in general, be second order ODEs in the time. If the first set of equations cannot be solved uniquely for p, then we have a system with constraints which is dealt with separately in section 2.1.5.

The equations of motion (2.1) can be re-expressed in a coordinate free manner, employing the

[1]The term 'trajectory' is employed in this thesis for paths which obey the equations of motion. The term 'path' is employed for more general paths in phase space.

notation of forms in differential geometry, as a condition on the velocity vector X_H of any trajectory in phase space:

$$\mathrm{d}H \;\approx\; X_H \lrcorner \Omega \tag{2.2}$$

where $X_H = \dot{q}^i \left(\frac{\partial}{\partial q^i}\right) + \dot{p}_i \left(\frac{\partial}{\partial p_i}\right)$, the tangent velocity vector at any point on the path. The symplectic form $\Omega := \mathrm{d}q^i \wedge \mathrm{d}p_i$ is a special 2-form defined on phase space, and d is the exterior derivative of forms on the phase-space manifold. Both $\mathrm{d}H$ and Ω are defined on the whole of phase space and independent of paths.

The left hand side is, in canonical coordinates:

$$\mathrm{d}H \;=\; \left(\frac{\partial H}{\partial q^i}\right)\mathrm{d}q^i + \left(\frac{\partial H}{\partial p_i}\right)\mathrm{d}p_i \tag{2.3}$$

and the right hand side is:

$$X_H \lrcorner \Omega \;=\; \left(\dot{q}^i \left(\frac{\partial}{\partial q^i}\right) + \dot{p}_i \left(\frac{\partial}{\partial p_i}\right)\right) \lrcorner \left(\mathrm{d}q^i \wedge \mathrm{d}p_i\right) = \; -\dot{p}_i dq^i + \dot{q}^i dp_i \tag{2.4}$$

so (2.2) is

$$\mathrm{d}H \;=\; \left(\frac{\partial H}{\partial q^i}\right)\mathrm{d}q^i + \left(\frac{\partial H}{\partial p_i}\right)\mathrm{d}p_i \;\approx\; -\dot{p}_i \mathrm{d}q^i + \dot{q}^i \mathrm{d}p_i \;=\; X_H \lrcorner \Omega \tag{2.5}$$

where here it can be seen that the coefficients of the components of the form in (2.2) are the Hamilton's equations (2.1) above.

2.1.1 Symplectic manifolds

We now express the objects and equations in the previous section geometrically, in the formalism of differential geometry, and generalize. An introductory text is the book by Cannas da Silva [25].

Above, $X_H = \dot{q}^i \left(\frac{\partial}{\partial q^i}\right) + \dot{p}_i \left(\frac{\partial}{\partial p_i}\right)$ is the vector field on phase space $\mathfrak{T}^*Q$, tangent to the dynamical trajectories in phase space, with $\left(\frac{\partial}{\partial q^i}\right), \left(\frac{\partial}{\partial p_i}\right)$ the coordinate basis vectors $\vec{e}_i, \vec{e}^i$ respectively, $\mathrm{d}H$ is the exterior derivative of the 0-form (function) H on $\mathfrak{T}^*Q$ and

$$\Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i = -\mathrm{d}(p_i \mathrm{d}q^i) = -\mathrm{d}\Theta \tag{2.6}$$

is a symplectic form on $\mathfrak{T}^*Q$, constructed from Θ, which is the *canonical tautological 1-form* of the dual tangent bundle. A tautological form on a form bundle is a form defined at each point on the bundle which is equal to that point on the fiber viewed as a form on the base space point projection of that same fiber and which is zero when contracted with vertical vectors. The convention of the minus sign factor has not been fixed and depends on the author. The convention above is adhered to throughout this thesis.

The structure $(\mathfrak{T}^*Q, \Omega)$ can be usefully generalized with the following definition:

A *symplectic manifold* is a manifold $\mathbb{M}$ with a form Ω, called a symplectic form, which is a closed ($\mathrm{d}\Omega = 0$) and non-degenerate 2-form. (These properties hold for $\Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$ above). A symplectic form is non-degenerate - which means that $X \lrcorner \Omega = \Omega(X) = 0$ implies $X = 0$ for any vector X at any point in the manifold. By Darboux's theorem, any point in a symplectic manifold is inside an open region which can be given local coordinates ('Darboux coordinates') (q^i, p_i), in which the symplectic form is $\Omega = \mathrm{d}q^i \wedge \mathrm{d}p_i$ throughout the region. Because it is non-degenerate, any symplectic form defines a map, at any point on the manifold, from vectors to 1-forms, $\flat \ : \ \mathfrak{X}(\mathbb{M}) \longrightarrow \Omega^1(\mathbb{M})$, which is invertible, $\sharp \ : \ \Omega^1(\mathbb{M}) \longrightarrow \mathfrak{X}(\mathbb{M})$ - thus for every function $f : \mathbb{M} \to \mathbb{R}$ on a manifold $\mathbb{M}$ with a symplectic form Ω, there is a unique vector field, called a *hamiltonian vector field*, $X = (\mathrm{d}f)^\sharp := \Omega^{-1} \lrcorner \mathrm{d}f$ such that $\mathrm{d}f = X_f \lrcorner \Omega$. In particular, any function f on the symplectic manifold $\mathbb{M}$ thus generates a vector field which can be integrated to give parametrized curves which foliate the symplectic manifold and which can be viewed as a parametrized flow, called the *hamiltonian flow* generated by f. In Hamiltonian mechanics, a classical dynamical system is specified by the phase space, together with a specific function (observable) on phase space called the Hamiltonian of the system and these curves are the evolution of the classical system in phase space, with the flow parametrization being the time, and $f = H$, the Hamiltonian. The flow X_f generated by f is a *symplectomorphism* (a diffeomorphism which preserves the symplectic form) because

$$\mathcal{L}_{X_f}\Omega = X_f \lrcorner \mathrm{d}\Omega + \mathrm{d}(X_f \lrcorner \Omega) = X_f \lrcorner 0 + \mathrm{dd}f = 0 \tag{2.7}$$

where $\mathcal{L}_{X_f}$ is the Lie derivative of flow of the vector field X_f, and $\mathrm{d}\Omega = 0$, due to the symplectic form being a closed form, and also $\mathrm{dd} = 0$ for the exterior derivative of forms. The space of infinitesimal hamiltonian symplectomorphisms $\mathfrak{ham}(\mathbb{M})$ is isomorphic to the space of functions on $\mathbb{M}$ mod the constant functions: $\mathfrak{ham}(\mathbb{M}) \simeq C^\infty(\mathbb{M})/\mathbb{R}$.

More generally, an infinitesimal symplectomorphism X requires that $\mathcal{L}_X\Omega = X \lrcorner \mathrm{d}\Omega + \mathrm{d}(X \lrcorner \Omega) = 0 + \mathrm{d}(X \lrcorner \Omega) = 0$, which shows that the 1-form $X^\flat := X \lrcorner \Omega$ dual to X must be closed. Moreover any closed 1-form θ may be used to generate such a symplectomorphism via the non-degeneracy of Ω: $X = \theta^\sharp$. Thus the space of infinitesimal symplectomorphisms $\mathfrak{sym}(\mathbb{M})$ is isomorphic to the space of closed 1-forms on $\mathbb{M}$: $\mathfrak{sym}(\mathbb{M}) \simeq \ker_{\Omega^1(\mathbb{M})}(\mathrm{d}) =: \Omega^1_{closed}(\mathbb{M})$, and exact 1-forms generate infinitesimal hamiltonian symplectomorphisms: $\mathfrak{ham}(\mathbb{M}) \simeq \mathrm{im}_{C^\infty(\mathbb{M})}(\mathrm{d}) =: \Omega^1_{exact}(\mathbb{M})$.

The space of infinitesimal hamiltonian symplectomorphisms $\mathfrak{ham}(\mathbb{M})$ is a Lie algebra with the Lie brackets of vector fields and is a Lie subalgebra of the space of infinitesimal symplectomorphisms $\mathfrak{sym}(\mathbb{M})$.

There is a short exact sequence

$$0 \longrightarrow H^0_{dR}(\mathbb{M}) \xrightarrow{i} C^\infty(\mathbb{M}) \xrightarrow{\sharp\circ\mathrm{d}} \mathfrak{sym}(\mathbb{M}) \xrightarrow{\flat} H^1_{dR}(\mathbb{M}) \longrightarrow 0 \tag{2.8}$$

In addition, because it is non-degenerate, a symplectic form can be inverted to a Poisson bivector: $\Pi := \Omega^{-1} = \left(\frac{\partial}{\partial q^i}\right) \wedge \left(\frac{\partial}{\partial p_i}\right)$, in local Darboux coordinates, and one can define a Poisson bracket on functions on a symplectic manifold $\mathbb{M}$ by

$$\{f, g\} := -\mathrm{d}f \llcorner \Pi \lrcorner \mathrm{d}g = f \cdot \left(\frac{\overleftarrow{\partial}}{\partial q^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i} \right) \cdot g = \left(\frac{\partial f}{\partial q^i}\right)\left(\frac{\partial g}{\partial p_i}\right) - \left(\frac{\partial f}{\partial p_i}\right)\left(\frac{\partial g}{\partial q^i}\right) \tag{2.9}$$

in local Darboux coordinates.

In the study of symmetries of a dynamical system, one is usually interested in some particular Lie subalgebra (often under which some particular dynamical Hamitonian function is invariant) of the Lie algebra of infintesimal symplectomorphisms. In particular, in a Lie subalgebra of the Lie algebra of hamiltonian symplectomorphisms $\mathfrak{ham}(\mathbb{M})$, where the isomorphism $\sharp \circ \mathrm{d}$ with a subspace of $C^\infty(\mathbb{M})/\mathbb{R}$ is a Lie algebra map with the Poisson bracket on $C^\infty(\mathbb{M})/\mathbb{R}$. Such a Lie algebra is called a *Poisson action* on $\mathbb{M}$.

2.1.2 Poisson brackets

A *Poisson bracket* is a binary bilinear operation defined on an associative ring R over a field $\mathfrak{K}$, which is antisymmetric,

$$\{f, g\} = -\{g, f\} \tag{2.10}$$

and obeys the Jacobi identity,

$$\{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = 0 \tag{2.11}$$

and the Leibnitz rule on the associative product,

$$\{fg, h\} = f\{g, h\} + \{f, h\}g \tag{2.12}$$

for all elements $f, g, h \in R$.

The ring R together with the Poisson bracket is called a *Poisson algebra*. The Leibnitz rule implies that, for any element h, $\{h, \cdot\}$ and $\{\cdot, h\}$ are derivations on R.

Of particular interest here, the multiplicative ring of functions $C^\infty(\mathbb{M})$ over the field $\mathbb{R}$ or $\mathbb{C}$ on a symplectic manifold $\mathbb{M}$, has a canonical Poisson bracket.

The ring of functions $C^\infty(\mathbb{M})$ on a symplectic manifold $(\mathbb{M}, \Omega)$ is a Poisson algebra where the canonical Poisson bracket is defined by

$$\{f, g\} = X_f \lrcorner \Omega \llcorner X_g = X_f \lrcorner X_g \lrcorner \Omega = \Omega(X_f, X_g) = \mathrm{d}f \llcorner X_g = \mathrm{d}f \llcorner (\mathrm{d}g)^\flat = X_f \lrcorner \mathrm{d}g = (\mathrm{d}f)^\flat \lrcorner \mathrm{d}g \tag{2.13}$$

Note that Leibnitz rule implies that, for any function $h \in C^\infty(\mathbb{M})$, $X_h \cdot := (\mathrm{d}h)^\flat \cdot = \{h, \cdot\}$ is a derivation of functions on $\mathbb{M}$. It is also, as shown in the previous section, a hamiltonian symplectomorphism of $\mathbb{M}$.

For a symplectic manifold $\mathbb{M}$, the canonical Poisson bracket can be expressed as a bivector field, Π where $\Pi := \left(\frac{\overleftarrow{\partial}}{\partial q^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i} \right)$ locally, where (q, p) are local Darboux coordinates.

For the Poisson bivector field Π, which is the inverse of Ω, viewing the latter as a map from vector fields to 1-form fields on the symplectic manifold, the Jacobi identity is equivalent to the Shouten bracket identity $\{\Pi, \Pi\}_{Sh} = 0$, which is an integrability condition on Π, equivalent to the closure of the symplectic form [61]. The set of C^∞ functions on a symplectic manifold form a Poisson algebra, which is a powerful structure for analyzing dynamical systems as well as for canonical quantization.

Employing the corresponding canonical Poisson bracket, $dH = X_H \lrcorner \Omega$ can be inverted as $\{\cdot, H\} = X_H$, and it is readily seen that $\{\cdot, f\} = X_f$ is a derivation said to be generated by f, for any function f on a symplectic manifold. The equation of motion for the rate of change of an observable $O = O(q^i, p_i)$ can be written succinctly with Poisson brackets: $\dot{O} \approx \{O, H\}$. If the observable depends explicitly on time, $O = O(q^i, p_i, t)$, then the rate of change is given by $\dot{O} \approx \{O, H\} + \frac{\partial O}{\partial t}$. In the particular cases where the observable is a Darboux coordinate, $O = q^i$ and $O = p_i$, $\dot{q}^i \approx \{q^i, H\}$ and $\dot{p}_i \approx \{p_i, H\}$ are Hamilton's equations of motion (2.1), if the Poisson brackets are expressed in the same Darboux coordinates (2.9). In the special case when the observable is the Hamiltonian function, we have $\dot{H} \approx \{H, H\} = 0$, because of the anti-symmetry of the Poisson bracket. This results in the fact that any function is constant along the trajectories it generates: $X_f(f) = \{f, f\} = 0$. If a function f Poisson commutes with H, that is: $\{f, H\} = 0$, then it is a constant of motion: $\dot{f} \approx \{f, H\} = 0$, and vice versa. If two functions f and g both Poisson commute with H, then $\{f, g\}$ and fg also Poisson commute with H, because of the Jacobi identity, and the Leibnitz property respectively. Thus, the set of functions which Poisson commute with H form a sub-algebra of the Poisson algebra of functions on $\mathcal{M}$.

As well as the time evolution explained above, a hamiltonian flow may represent a symmetry or other transformation which is a global symplectomorphism.

2.1.3 Symplectic geometry

The definition of a symplectic form may be weakened: a *presymplectic* form is a closed 2-form which is not necessarily non-degenerate, and an *almost symplectic* form is a non-degenerate 2-form which is not necessarily closed. Submanifolds of symplectic manifolds inherit a presymplectic form from the symplectic form of the ambient manifold via the pull back of the embedding map. A submanifold is symplectic, coisotropic, isotropic, lagrangian, if the kernel of the inherited presymplectic form (viewed as a map from vector fields in the submanifold to form fields in the submanifold defined on the submanifold) is 0, is in the tangent space of the submanifold, includes the tangent space, is the tangent space, respectively. A lagrangian submanifold has the maximum dimensionality for an isotropic submanifold, which is half that of the ambient symplectic manifold. The notion of lagrangian submanifold is important in symplectic geometry. Coisotropic submanifolds are an ingredient in Marsden-Weinstein reduction.

Because of Darboux's theorem, all symplectic manifolds of the same dimensionality are locally the same, and there is a large group of symplectomorphisms. This is different from manifolds with metric, where local properties such as curvature can be defined and where the group of isometries is usually trivial.

The set of symplectic manifolds together with maps which are symplectomorphisms form a category [23].

2.1.4 Time dependent Hamiltonians and extended phase space

If the Hamiltonian function on phase space depends explicitly on time, the results of the previous section hold, except when considering the foliation of the phase space generated by H. In this case the space considered should be the bundle $\mathfrak{T}^*Q \times \mathbb{R}$ of phase-space fibers over time base space $\mathbb{R}$. Each fiber $\mathfrak{T}^*Q$ represents the phase space at a particular time. The Hamiltonian may no longer be a constant of motion: $\dot{H} \approx \{H, H\} + \frac{\partial H}{\partial t} = \frac{\partial H}{\partial t}$. However the rate of change of a time dependent observable is given by the same equation as above:

$$\dot{O} \approx \{O, H\} + \frac{\partial O}{\partial t} \tag{2.14}$$

Another symplectic manifold is usefully employed in classical mechanics: *the extended phase space*, $\mathfrak{T}^*\widetilde{Q} := \mathfrak{T}^*Q \times \mathbb{R}^2$, which has, in addition to the coordinates (q^i, p_i) on the phase space, an extra canonical pair of coordinates: time and energy coordinates, t, s. The symplectic form is $\widetilde{\Omega} = dq^i \wedge dp_i + dt \wedge ds$ and the corresponding Poisson bracket is

$$\{f, g\}_{\mathfrak{T}^*\widetilde{Q}} = -\mathrm{d}f \llcorner \widetilde{\Pi} \lrcorner \mathrm{d}\widetilde{g} = f \cdot \left(\frac{\overleftarrow{\partial}}{\partial q^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i} \right) \cdot \widetilde{g} + f \cdot \left(\frac{\overleftarrow{\partial}}{\partial t} \wedge \frac{\overrightarrow{\partial}}{\partial s} \right) \cdot \widetilde{g} \tag{2.15}$$

We define the extended Hamiltonian as $\widetilde{H}(q,p,t,s) = H(q,p,t) + s$, then the symplectic form $\widetilde{\Omega}$ pulled back to the codimension 1 hypersurface $\bar{H}$ embedded in the extended phase space: $i : \bar{H} \longrightarrow \mathfrak{T}^*\widetilde{Q}$, defined by the level set of $\widetilde{H} = 0$ inside $\mathfrak{T}^*\widetilde{Q}$, $\widetilde{\Omega}_{\bar{H}} = i^*\widetilde{\Omega}$, is a (degenerate) presymplectic (i.e. closed and degenerate) 2-form. This 2-form has a one dimensional characteristic distribution, that is, at any point of $\bar{H}$ the space of vectors V such that $\widetilde{\Omega}_{\widetilde{H}}$ annihilates V , $V \lrcorner \widetilde{\Omega}_{\widetilde{H}} = 0$, is one dimensional and, in addition, tangent to $\bar{H}$. This defines a foliation of the space $\bar{H}$ into unparametrized curves (1-dimensional surfaces embedded in $\bar{H}$).

This defines the same system as in (2.1), with the time evolution of the system being given by movement along these curves through the extended phase space. The time is not given by parametrization of the curves, which are not parametrized in this case - unlike the curves on the phase space above - but by the t coordinate of the curve in the extended phase-space coordinate system.

We use the extended phase space in the same way as the phase space to obtain rates of change of observables. By extending a time varying observable $O = O(q,p;t)$ on $\mathfrak{T}^*Q$ to $O(q,p,t,s) = O(q,p;t)$ on $\mathfrak{T}^*\widetilde{Q}$, an then taking the Poisson brackets with the extended Hamiltonian, we have

$$\begin{aligned}
\dot{O} \approx \{O, \widetilde{H}\}_{\mathfrak{T}^*\widetilde{Q}} &= -\mathrm{d}O \llcorner \widetilde{\Pi} \lrcorner \mathrm{d}\widetilde{H} = O \cdot \left(\frac{\overleftarrow{\partial}}{\partial q^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i} \right) \cdot \widetilde{H} + O \cdot \left(\frac{\overleftarrow{\partial}}{\partial t} \wedge \frac{\overrightarrow{\partial}}{\partial s} \right) \cdot \widetilde{H} = \\
&= \left(\frac{\partial O}{\partial q^i}\right)\left(\frac{\partial H}{\partial p_i}\right) - \left(\frac{\partial O}{\partial p_i}\right)\left(\frac{\partial H}{\partial q^i}\right) + \left(\frac{\partial O}{\partial t}\right)\left(\frac{\partial s}{\partial s}\right) + 0\left(\frac{\partial H}{\partial t}\right) \\
&= \{O, H\}_{\mathfrak{T}^*Q} + \left(\frac{\partial O}{\partial t}\right)
\end{aligned} \tag{2.16}$$

which is the same as (2.14) above for time dependent observables and Hamiltonians.

The information previously given by the Hamiltonian function on phase space is provided in the extended phase space by the codimension 1 constraint, $\widetilde{H} = 0$, on the extended phase space. Any such regular hypersurface $\bar{H}$ on any symplectic manifold foliates the hypersurface $\bar{H}$ into curves through the characteristic distribution (which lies inside the tangent space of the hypersurface) of the symplectic form pulled back to the hypersurface by the embedding map, which becomes a presymplectic form whose degenerate vectors define the characteristic distribution.

This is special case of a more general procedure, given in detail below in section (2.4.2), known as 'Marsden-Weinstein reduction', where constraints on the dynamical variables and momenta in the phase space (symplectic manifold) $\mathcal{M}$ (with coordinates z) defined by functions,

$C^k(z) = 0, k = 1 \ldots K$, on the manifold, result in a constraint surface $\overline{C}$ (of codimension K if the functions C^k are regular on $\overline{C}$) embedded in the symplectic manifold, $i : \overline{C} \hookrightarrow \mathbb{M}$. Under certain conditions, the characteristic tangent space at each point in the surface relative to the symplectic form foliates the constraint surface into leaves, and the set of leaves is a manifold, $\mathcal{M}//C$, called *the reduced phase space*, with a symplectic form, Ω_C. This symplectic form comes from the presymplectic form which is the pullback by the embedding map of the original symplectic form. It inherits the closure condition and is invertible because the kernel of the presymplectic form, which is the characteristic distribution, is mod-ed out in the construction, leaving a symplectic form. For instance, symmetries in the Lagrangian of dynamical system become constraints in the phase space under the Legendre transformation and the reduced phase space is the physical dynamical system with the non-dynamical degrees of freedom, which are the symmetry transformations, removed.

In the special case under consideration here, the codimension 1 constraint, $\widetilde{H} = 0$, on the extended phase space $\mathfrak{T}^*\widetilde{Q}$, generates the 'symmetry transformation' symplectomorphism $\{\cdot, \widetilde{H}\}_{\mathfrak{T}^*\widetilde{Q}}$ which is the time evolution.

In the above, the symplectic form $\widetilde{\Omega} = \mathrm{d}q^i \wedge \mathrm{d}p_i + \mathrm{d}t \wedge \mathrm{d}s$ on the extended phase space $\mathfrak{T}^*\widetilde{Q}$ reduces, under Marsden-Weinstein reduction, to $\Omega_C = \mathrm{d}q^i \wedge \mathrm{d}p_i$ on the phase space $\mathfrak{T}^*Q$, which is the reduced phase space. The equations of motion are given by

$$\begin{aligned} 0 \approx X_{\bar{H}} \lrcorner\, i^*\widetilde{\Omega} = (\dot{q}^i \left(\frac{\partial}{\partial q^i}\right) + \dot{p}_i \left(\frac{\partial}{\partial p_i}\right) + \dot{t} \left(\frac{\partial}{\partial t}\right) + \dot{s} \left(\frac{\partial}{\partial s}\right)) \lrcorner\, (\mathrm{d}q^i \wedge \mathrm{d}p_i + \mathrm{d}t \wedge \mathrm{d}(-H)) = \\ = \dot{q}^i \mathrm{d}p_i - \dot{p}_i \mathrm{d}q^i - \dot{t} \left(\frac{\partial H}{\partial p_i}\right) \mathrm{d}p_i - \dot{t} \left(\frac{\partial H}{\partial q^i}\right) \mathrm{d}q^i = X_H \lrcorner\, \Omega - \ \dot{t}\, \mathrm{d}H \end{aligned} \tag{2.17}$$

where $X_{\bar{H}}$ is constrained to lie in the hypersurface $\bar{H}$ given by $\widetilde{H} = H(q, p, t) + s = 0$. The magnitude of the vectors in the characteristic distribution is not specified so we are free to choose $\dot{t} = 1$. What we have shown in (2.17) is that the equation for the vector field $X_{\bar{H}}$ which is characteristic distribution of $\bar{H}$ are Hamilton's equations of motion for H, in the notation of differential geometry of symplectic manifolds.

The notion of putting energy on the same footing as momentum, and time on the same footing as q is reminiscent of special relativity. The dynamics can be pictured in a timeless geometrical way, where time is canonically paired with energy. Time is defined by the notion that H is the energy of the system (so the energy coordinate $s = -H$).

2.1.5 Hamiltonian systems with constraints

The *regular form* of the Hamiltonian function is a convex function of the momenta, which allows the first Hamilton equation to be inverted, so that the momenta can be expressed as a function of the time derivatives of the configuration variables. Often the Hamiltonian is quadratic in the momenta. If the Hamiltonian is linear in a momentum then there is a constraint, in the sense that the corresponding velocity is a function of the other momenta (hence the other velocities) and configuration variables. This results, for instance, in that the velocity cannot be arbitrarily set as an initial condition. The latter is an example of an irregular form of the Hamiltonian function. The consequence of irregularity is that only part of the phase space has non-zero solutions, i.e. where the trajectory is more than a single point, or that one considers equivalence classes of trajectories to correspond to individual physical solutions.

One may consider constraints where motion is restricted to a submanifold of the phase space. One may impose a constraint from the beginning in the form of a submanifold of the symplectic manifold. This is called a primary constraint, and often occurs when the Legendre transformation is not invertible, which will occur if there is a symmetry in the action. A given Hamiltonian may only be consistent with the primary constraint (i.e. the hamiltonian vector field is tangent to the constraint surface) on a submanifold of the primary constraint submanifold, leading to a secondary constraint. Similarly, consistency with the secondary constraint may lead to a tertiary constraint, and so on until the constaint surface is small enough to be consistent with the Hamiltonian.

The consequences of symmetries are an important topic and examined in section 2.4.

2.2 Lagrangian mechanics

The configuration space equations of motion can be obtained as Euler-Lagrange equations for a variational principle, namely where an integral (over time), (known as the action of the path), of a function, denoted the Lagrangian, of the configuration dynamical variables and of their first and higher time derivatives, is stationary with respect to small variations of path. The variations are such that the configuration variables and the time are fixed (i.e. cannot be varied) at the endpoints of the path.

Here we will only consider Lagrangians which are functions of first time derivatives (velocities) and not higher derivatives of the configuration variables, which are physically less relevant.

The physical system is defined by a *Lagrangian*, which is a function of time, the configuration dynamical variables, and the first time-derivatives of the dynamical variables (we will not consider Lagrangians with higher time-derivatives). Starting from the Lagrangian, $L = L(q^i, v^i, t)$ on the time extended velocity phase space $\mathfrak{T}Q \times \mathbb{R}$, where q^i, v^i are locally adapted coordinates on $\mathfrak{T}Q$, which is the tangent bundle of Q, and t is time, the *action* or *configuration space action* of a time parametrized path $q^i(t)$ in configuration space, from time $t = t_0$ to time $t = t_1$ is defined to be:

$$S[q^i(t)] \; = \int_{t_0}^{t_1} \; L(q^i(t), \partial_t q^i(t), t) \; \; \mathrm{d}t \tag{2.18}$$

where $\partial_t q^i$ is the time derivative of the functions $q^i(t)$ defining the path. If we vary the path (but not time t) $C(\mathfrak{T}Q) = (q^i(t), \dot{q}^i(t))$ in the velocity phase space by the infinitesimal variation $\delta q^j(t)$, the infinitesimal variation in the action is

$$\begin{aligned}\delta S[q^i(t)] \; &= \; \delta \int_{C(\mathfrak{T}Q)} \; L(q^i, \partial_t q^i, t) \; \; \mathrm{d}t = \int_{C(\mathfrak{T}Q)} \; \delta L(q^i, \partial_t q^i, t) \; \; \mathrm{d}t \\ &= \int_{C(\mathfrak{T}Q), v^i = \dot{q}^i} \Big[- \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial v^j} \right) - \left(\frac{\partial L}{\partial q^j} \right) \right\} \delta q^j \; + \; \left\{ \frac{d}{dt} [\left(\frac{\partial L}{\partial v^j} \right) \delta q^j] \right\} \Big] \mathrm{d}t \\ &= \int_{C(\mathfrak{T}Q), v^i = \dot{q}^i} - \left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial v^j} \right) - \left(\frac{\partial L}{\partial q^j} \right) \right\} \delta q^j \; \; \mathrm{d}t \; + \; \left\{ \left(\frac{\partial L}{\partial v^j} \right) \delta q^j \right\}_{\partial C(\mathfrak{T}Q)}\end{aligned} \tag{2.19}$$

where $\partial C(\mathfrak{T}Q)$ are the endpoints of the path $C(\mathfrak{T}Q)$.

Then a stationary point of the action, $\delta S[q^i(t)] \; = \; 0$, for paths $q^i(t)$, where the end points are fixed ($\delta q^j = 0$ at $\partial C(\mathfrak{T}Q)$) occurs when the integrand in the last equality is zero for all variations $\delta q^j(t)$, and this occurs when the path is such that the following coefficients of δq^j are zero:

$$\left\{ \frac{d}{dt} \left(\frac{\partial L}{\partial v^j} \right) - \left(\frac{\partial L}{\partial q^j} \right) \right\}_{v^i = \dot{q}^i} \approx 0, \qquad j = 1 \ldots N. \tag{2.20}$$

These are the Euler-Lagrange equations for L, and are the equations of motion which are satisfied on a physical trajectory, $q^i_{phy}(t)$ for a dynamical system specified by a Lagrangian L. The action functional $S[]$ is at a stationary point for such a solution: $\delta S[q^i_{phy}(t)] \; = \; 0$.

From the Lagrangian function $L = L(q^i, v^i, t)$ on the velocity phase space $\mathfrak{T}Q$, where q^i, v^i are locally adapted coordinates on $\mathfrak{T}Q$, which is the tangent bundle of Q, and t is time, we can construct a closed (presymplectic) 2-form

$$\Omega_L := -\mathrm{d}\Theta_L := -\mathrm{d}(P_i \mathrm{d}q^i) = -\mathrm{d}P_i \wedge \mathrm{d}q^i \tag{2.21}$$

where $P_i \; := \; \left(\frac{\partial L}{\partial v^i} \right)$, and a velocity Hamiltonian function $H_L(q^i, v^i, t) \; := \; v^i P_i - L$ on $\mathfrak{T}Q$, which are both geometrical objects covariant under coordinate changes. Then, the 'geometric Euler-Lagrange equation',

$$X \lrcorner \, \Omega_L - \mathrm{d}H_L \approx 0 \tag{2.22}$$

where $X = \dot{q}^i \left(\frac{\partial}{\partial q^i}\right) + \ddot{q}^i \left(\frac{\partial}{\partial v^i}\right)$, a vector field on $\mathfrak{T}Q$, is

$$X \lrcorner \Omega_L - \mathrm{d}H_L = -\left\{\ddot{q}^i \frac{\partial^2 L}{\partial v^i \partial v^j} + \dot{q}^i \frac{\partial^2 L}{\partial q^i \partial v^j} - \left(\frac{\partial L}{\partial q^j}\right)\right\}_{v^i = \dot{q}^i} \mathrm{d}q^j =$$

$$= -\left\{\frac{d}{dt}\left(\frac{\partial L}{\partial v^j}\right) - \left(\frac{\partial L}{\partial q^j}\right)\right\}_{v^i = \dot{q}^i} \mathrm{d}q^j \approx 0 \qquad (2.23)$$

if $v^i = \dot{q}^i$. The equation at the right hand end above can be seen to be the Euler-Lagrange equations (2.20) above for the variational problem $\delta S[q^i(t)] = \delta \int L \, \mathrm{d}t = 0$, with the time t and the values of the q^i fixed at the end points of the integral. Thus the Euler-Lagrange equations are equivalent to the 'geometric Euler-Lagrange equation', $X \lrcorner \Omega_L - dH_L \approx 0$, when the Lagrangian is regular: $Det\left\{\left(\frac{\partial^2 L}{\partial v^i \partial v^j}\right)\right\} \neq 0$.

2.2.1 Relation between Lagrangian and Hamiltonian mechanics

The relation between Lagrangian mechanics and Hamiltonian mechanics is via the Legendre transformation $\mathbb{F}\mathcal{L}$, which is a canonical map, generated by the Lagrangian function, between the extended velocity phase space $\mathfrak{T}\widetilde{Q}$ and the time-extended phase space $\mathfrak{T}^*Q' := \mathbb{R} \times \mathfrak{T}^*Q$:

$$\mathbb{F}\mathcal{L} : \mathfrak{T}\widetilde{Q} \longrightarrow \mathfrak{T}^*Q' :: (t, q^i, v^i) \longmapsto (t, q^i, p_i) = (t, q^i, P_i), \qquad (2.24)$$

where t, q^i, p_i are locally adapted canonical coordinates on $\mathfrak{T}^*Q'$. If the Lagrangian is regular, the map is a bundle isomorphism between $\mathfrak{T}\widetilde{Q}$ and $\mathfrak{T}^*Q'$, otherwise the image of $\mathfrak{T}\widetilde{Q}$ is called the primary constraint surface $P_{\mathcal{L}} \subset \mathfrak{T}^*Q'$. There is also a Legendre transformation between the extended velocity phase space $\mathfrak{T}\widetilde{Q}$ and the extended phase space $\mathfrak{T}^*\widetilde{Q}$:

$$\mathbb{F}\tilde{\mathcal{L}} : \mathfrak{T}\widetilde{Q} \longrightarrow \mathfrak{T}^*\widetilde{Q} :: (t, q^i, v^i) \longmapsto (t, q^i, s, p_i) = (t, q^i, v^i P_i - L \ , P_i) = (t, q^i, H_L \ , P_i) \qquad (2.25)$$

The velocity Hamiltonian function $H_L(q^i, v^i, t) := v^i P_i - L$, on the extended velocity phase space $\mathfrak{T}\widetilde{Q}$, is the pull back via the Legendre transform $\mathbb{F}\mathcal{L}$ of the phase-space Hamiltonian $H(t, q^i, p_i)$ on the primary constraint surface $P_{\mathcal{L}} \subset \mathfrak{T}^*Q'$. The Hamiltonian system $(P_{\mathcal{L}} \subset \mathfrak{T}^*Q', \Omega, H)$ is pulled back via the Legendre transform to $\mathbb{F}\mathcal{L}$ to $(\mathfrak{T}\widetilde{Q}, \Omega_L, H_L)$. (Here the presymplectic and symplectic forms Ω_L and Ω act respectively on the fibers $\mathfrak{T}Q$ and $\mathfrak{T}^*Q$ of the bundles $\mathfrak{T}\widetilde{Q}$ and $\mathfrak{T}^*Q'$ over time $\mathbb{R}$). The geometric Euler-Lagrange equation 2.22 $X \lrcorner \Omega_L - \mathrm{d}H_L = 0$ is the pull back of the (geometrically expressed) Hamilton's equations $X_p \lrcorner \Omega - \mathrm{d}H = 0$, if the vector field X_p can be pulled back to $\mathfrak{T}\widetilde{Q}$, which occurs when the Legendre transformation is regular and the map is 1-1.

When the Legendre transformation is not invertible, the velocity phase-space Hamiltonian H_L will not define a Hamiltonian H on all of $\mathfrak{T}^*Q'$, but only on a submanifold $P_{\mathcal{L}}$.

The 2-form Ω_L on the velocity phase space, defined above, is the pullback via the Legendre transform of the symplectic form on the phase space and extended phase space: $\mathbb{F}\mathcal{L}^*\Omega = \Omega_L = \mathbb{F}\tilde{\mathcal{L}}^*\tilde{\Omega}$. In fact, the action S of a path $C_{\mathfrak{T}Q} = (q^i(t), \dot{q}^i(t))$ in $\mathfrak{T}Q$ is the integral of the extended canonical 1-form, $\widetilde{\Theta} = p_i \mathrm{d}q^i + s\mathrm{d}t$ locally, over the constrained path $C_{\bar{H}} = \mathbb{F}\tilde{\mathcal{L}}(C_{\mathfrak{T}\widetilde{Q}})$ inside the hypersurface $\bar{H}$ embedded in the extended phase space, and of the 1-form $p_i\mathrm{d}q^i - H\mathrm{d}t$ over the path $C_{\mathfrak{T}^*Q'} = F\mathcal{L}(C_{\mathfrak{T}Q})$ in time-extended phase space, and of the 1-form $\mathrm{d}L = \left(\frac{\partial L}{\partial v^i}\right)\mathrm{d}v^i + \left(\frac{\partial L}{\partial q^i}\right)\mathrm{d}q^i + \left(\frac{\partial L}{\partial t}\right)\mathrm{d}t$ over the path $C_{\mathfrak{T}\widetilde{Q}}$ in extended velocity phase space, and (by definition) of the 1-form $L\mathrm{d}t$ over the path $C_{Q'}$ in time-extended configuration space:

$$\int_{C_{\bar{H}}} \widetilde{\Theta} = \int (p_i \mathrm{d}q^i + s\mathrm{d}t) \llcorner \left(\frac{\partial}{\partial t}\right)\Big|_{C_{\bar{H}}} = \int_{C_{\mathfrak{T}^*Q'}} (p_i\dot{q}^i - H)\,\mathrm{d}t = \int_{C_{\mathfrak{T}\widetilde{Q}}} L\,\mathrm{d}t = S. \tag{2.26}$$

An infinitesimal variation $Y_{\bar{H}}$ of the path $C_{\bar{H}} = \mathbb{F}\tilde{\mathcal{L}}(C_{\mathfrak{T}\widetilde{Q}})$ inside $\bar{H}$ in extended phase space is

$$\begin{aligned}\delta_Y \int_{C_{\mathfrak{T}\widetilde{Q}}} L\,\mathrm{d}t &= \delta_{Y_{\bar{H}}} \int_{C(\bar{H})} \widetilde{\Theta} = \int_{C(\bar{H})} \mathcal{L}_{Y_{\bar{H}}} \widetilde{\Theta} = \int_{C(\bar{H})} (\mathrm{d}i_{Y_{\bar{H}}} + i_{Y_{\bar{H}}}\mathrm{d})\widetilde{\Theta} \\ &= \int_{\partial C(\bar{H})} i_{Y_{\bar{H}}} \widetilde{\Theta} + \int_{C(\bar{H})} i_{Y_{\bar{H}}} \widetilde{\Omega}\end{aligned} \tag{2.27}$$

If the variation vector field is zero, $Y_{\bar{H}} = 0$, on the boundary (end points) $\partial C(\bar{H})$ of the path $C(\bar{H})$, we obtain

$$\delta_Y \int_{C_{\mathfrak{T}\widetilde{Q}}} L\,\mathrm{d}t = \delta_{Y_{\bar{H}}} \int_{C(\bar{H})} \widetilde{\Theta} = \int_{C(\bar{H})} i_{Y_{\bar{H}}}\widetilde{\Omega} = - \int_{C(\bar{H})} X_{\bar{H}} \lrcorner\, Y_{\bar{H}} \lrcorner\, \widetilde{\Omega}\ \mathrm{d}t \tag{2.28}$$

where $X_{\bar{H}}$ is the tangent vector of the time-parametrized path in extended phase space.

If the symplectic equations of motion in extended phase space hold for the trajectory $C(\bar{H})$: $i_{X_{\bar{H}}}\widetilde{\Omega} \approx 0$, then the variational principle holds: the integrand is zero for any variation $Y_{\bar{H}}$ of the path with fixed endpoints:

$$\delta_Y \int_{C_{\mathfrak{T}Q}} L\,dt = \delta_{Y_{\bar{H}}} \int_{C(\bar{H})} \widetilde{\Theta} = \int_{C(\bar{H})} Y_{\bar{H}} \lrcorner\, (X_{\bar{H}} \lrcorner\, \widetilde{\Omega})\ \mathrm{d}t \approx 0 \tag{2.29}$$

Thus we have shown the immediate relation between the symplectic equations of motion and the variational principle.

2.2.2 Phase-space Lagrangian formalism

Hamiltonian mechanics can be re-expressed with Lagrangian formalism. This will be very useful both in phase-space and multiphase-space mechanics. It also appears naturally in path integrals in quantum mechanics.

An action, whose Euler-Lagrange equations are the Hamilton's equations of motion for a Hamiltonian $H(q^i, p_i, t)$, can be constructed for a Hamiltonian system, which is specified by configuration variables q^i, conjugate momenta p_i, and a Hamiltonian function $H(q, p, t)$. The Lagrangian for this system is

$$L_P(q^i, q^i_{,t}, p_i, t) := p_i q^i_{,t} - H(p_i, q^i, t) \tag{2.30}$$

where, in the action $S[q^i(t)] = \int L_P(q^i, q^i_{,t}, p_i, t)\ \mathrm{d}t$, $q^i_{,t} = \frac{\mathrm{d}q^i}{\mathrm{d}t}$ is the time derivative of q^i along the path $q^t(t)$.

This Lagrangian is called a *phase-space Lagrangian*, and this is called a first order formalism because the Euler-Lagrange equations have just first derivatives in time, which is the desirable property of Hamilton's equations. For there to be higher time derivatives, the Hamiltonian H would have to have at least quadratic terms in time derivatives of q or p.

The first order Lagrangian formalism employs a Lagrangian, $L_P(q^i, q^i_{,t}, p_i, t) = p_i\dot{q}^i - H(p_i, q^i, t)$, which is a function of canonical momenta as well as configuration variables and where H is the Hamiltonian function on the time extended phase space. The Euler-Lagrange equations for this Lagrangian for the variational problem where the configuration variables are fixed at the endpoints $\partial C(\mathfrak{T}^*Q)$ of the path $C(\mathfrak{T}^*Q)$ can be seen to be Hamilton's equations of motion, by considering the stationary point of the *phase-space action* with fixed end points:

$$\delta S_P[q^i(t), q^i_{,t}(t), p_i(t)] \ = \ \delta \int_{C(\mathfrak{T}^*Q)} L_P(q^i(t), q^i_{,t}(t), p_i(t), t) \ \ \mathrm{d}t =$$

$$\delta \int_{C(\mathfrak{T}^*Q)} (p_i\dot{q}^i - H)\mathrm{d}t = \int_{C(\mathfrak{T}^*Q)} \left\{ (\dot{q}^i - \left(\frac{\partial H}{\partial p_i}\right))\delta p_i - (\dot{p}_i + \left(\frac{\partial H}{\partial q^i}\right))\delta q^i \right\} \mathrm{d}t \ + \ [\delta q^i p_i]_{\partial C(\mathfrak{T}^*Q)} \tag{2.31}$$

It can be seen that if the path is such that the coefficients of δq^i and δp_i in the integrand above are zero, then the path is stationary for all infinitesimal variations where the end points are fixed. The coefficients would then be $\dot{q}^i - \left(\frac{\partial H}{\partial p_i}\right) \approx 0$ and $\dot{p}_i + \left(\frac{\partial H}{\partial q^i}\right) \approx 0$, which are the Hamilton's equations of motion.

To obtain the configuration space Lagrangian from a Hamiltonian, the momenta p_i in the first order action are substituted by expressions $p_i = P_i(q, \dot{q}, t)$ usually involving time derivatives of q^i obtained from the first Hamilton equation of motion, by solving it for the p_i. These can be solved if the Hamiltonian is regular.

The Legendre transformation of this latter action will reproduce the original Hamiltonian system:

$$L(q^i, q^i_{,t}, t) = L_P(q^i, q^i_{,t}, P_i(q^i, q^i_{,t}, t), t) \tag{2.32}$$

The first order formalism can sometimes be employed to combine the advantages of both Hamiltonian and Lagrangian formulations, and is frequently employed in this paper, in particular in the multiphase-space version.

2.3 Field theory

2.3.1 Lagrangian field theory

In applying Lagrangian mechanics to fields we, in the first instance, consider a field to be a mechanical system with an infinite number of degrees of freedom. These degrees of freedom are the field values at every point in space. For 4 dimensional Minkowski space we choose a time coordinate and the corresponding orthogonal Euclidean 3 space $\mathbb{R}^3$ and the configuration degrees of freedom are u^i_x where $i = 1 \dots N$ and $x \in \mathbb{R}^3$.

A first order Lagrangian field theory is based on the action of a field, which is a functional $S[u(x)] = \int L \; \mathrm{d}t$ of field configurations $u(x)$ on spacetime M^d, which for a local field is the integral over time of a Lagrangian L, which is a functional of all the field values $u^i(y,t)$ on a spatial slice $x^0 = t$ (where y represent the coordinates on the spatial slice), the first time derivatives $\partial_t u^i(y,t)$ of these field values on the spatial slice , the spatial derivatives of these field values $\partial^r_j u^i(y,t), r = 1 \dots d-1$ on the spatial slice, and the time t. For a local field, which is what will concern us here, the Lagrangian is the integral over a spatial slice of a Lagrangian density $\mathcal{L}$ which is a function of the spacetime point x as well as the field values u at the point, and of the first (for first order field theory) spatial and time derivatives of the field at the point. The spatial derivatives are also first order because we limit ourselves to relativistic fields where space and time are on the same footing.

For Minkowski space M^d with specified time t and space y^k coordinates, $M^d = \mathbb{R} \times \mathbb{R}^{d-1}$, field values $u^i(x)$ at point $x = (t, y^1, \dots, y^{d-1})$, Lagrangian L , Lagrangian density $\mathcal{L}$, the action is

$$S[u^i(x)] \; = \; \int_{\mathbb{R}} L\left(u^i(y^k,t), \frac{\mathrm{d}u^i(y^k,t)}{\mathrm{d}t}\right) \mathrm{d}t \; =$$

$$\int_{\mathbb{R}} \left\{ \int_{\mathbb{R}^{d-1}} \mathcal{L}\left(u^i(y^k,t), \frac{\partial u^i(y,t)}{\partial x^\mu}, y^k, t\right) \; \mathrm{d}^{d-1}y \right\} \mathrm{d}t = \int_{\mathbb{M}^d} \mathcal{L}\left(u^i(x), \frac{\partial u^i(x)}{\partial x^\mu}, x\right) \; \mathrm{d}^d x \quad (2.33)$$

Thus the action integral for a local field can be explicitly written in the formalism of differential geometry of metric manifolds, by the action being a spacetime integral of a local Lagrangian density $\mathcal{L}$ and the Lagrangian density being a spacetime scalar constructed from tensorial objects. The last integral above can be generalized from $\mathbb{M}^d$ to a general Lorentzian or Riemannian manifold. The action integral can be expressed in a geometrical way in the notation of tensors. The advantage of Lagrangian over Hamiltonian field theory is that the locality and spacetime covariance can be explicit. There are disadvantages: in particular the Euler-Lagrange equations for the action involve second time-derivatives of the fields.

Mathematical structure of field theory action

Information on the mathematical structure of classical field theories is found in [36]. Instead of starting, as in mechanics, from a bundle, $Q \times \mathbb{R}$, with fiber Q and base space $\mathbb{R}$ (for time), whose sections would be parametrized trajectories $q^i = q^i(t)$ in configuration space, the natural starting point for a field is the bundle $\mathcal{E}$ over space-time B, whose sections would be specific values of the fields defined over spacetime, $u^i = u^i(x^\mu)$, $a = 1 \ldots N$. The fiber at any point in spacetime are the possible field values at that point. Usually the fiber is a vector space or a manifold with extra structure as in sigma models.

The natural analogue of the time extended velocity phase space $\mathfrak{T}Q \times \mathbb{R}$ with local coordinates (q^i, v^i, t) is the *first jet bundle* (for first order field theories) which is the affine bundle of the first derivatives of sections of $\mathcal{E}$: $\pi^{\mathcal{E}, J^1\mathcal{E}} : J^1 E \longrightarrow \mathcal{E} :: (x^\mu, u^i, u^i_\mu) \mapsto (x^\mu, u^i)$, over the field configuration bundle $\pi^{B,\mathcal{E}} : \mathcal{E} \longrightarrow B :: (x^\mu, u^i) \mapsto (x^\mu)$, where B is the underlying d dimensional spacetime manifold. A spacetime field configuration $u(x^\mu)$ is given by a section of $\pi^{B,\mathcal{E}}$, $u : B \longrightarrow \mathcal{E}$, and has a prolongation to a section of $\pi^{B,J^1\mathcal{E}}$, $j^1 u : B \longrightarrow J^1\mathcal{E} :: (x^\mu) \longmapsto (x^\mu, u^i, u^i_\mu) = (x^\mu, u^i(x^\nu), \frac{\partial u^i(x^\mu)}{\partial x^\mu})$.

The analogue of the Lagrangian is the Lagrangian density, $\mathbb{L} : J^1\mathcal{E} \longrightarrow \Omega^d B$ which is a function (bundle map over B) on the total space of the first jet bundle , where $\Omega^d B$ is the bundle of d-forms on spacetime B. And the dynamical action, $S[u^i(x)]$, of the field is is given by the integral of the Lagrangian density over spacetime: $S[u^i(x)] = \int_B \mathbb{L}(j^1 u)$ as opposed to the action as the time integral of a Lagrangian.

Euler-Lagrange equations

The variational problem is to find the stationary point of the action in the space of paths in $U \subset B$ over variations of the field, $\delta_\xi u^i(x) = u^i(x) + \xi^i(x)$, where the variation is zero (i.e. u^i is kept fixed with values $u^i(y) = u^i_0(y)$ where $y \in \partial U$) on the boundary ∂U of the spacetime region U:

$$\delta_\xi S[(x)] = \delta_\xi \int_U \mathbb{L}(j^1 u) = \int_U \delta_\xi \mathbb{L}(j^1 u) \overset{!}{=} 0 \tag{2.34}$$

with ξ being an infinitesimal variation of the field configuration $u^i(x)$.

A path or configuration $u^i(x^\mu)$ in U which solves the variational problem (called a trajectory) satisfies the Euler-Lagrange equations

$$\partial_\mu \Big(\frac{\partial \mathcal{L}}{\partial u^i_\mu} (u^j(x^\nu), u^j_\nu(x^\lambda), x^\nu) \Big) - \frac{\partial \mathcal{L}}{\partial u^i} (u^j(x^\nu), u^j_\nu(x^\mu), x^\nu) \;\approx\; 0 \tag{2.35}$$

where $\mathbb{L} = \mathcal{L}\ \mathrm{d}^d x = \mathcal{L}\ \mathrm{d}x^0 \wedge \ldots \wedge \mathrm{d}x^{d-1}$, and $u^i_\nu(x^\mu) \approx \frac{\partial u^i}{\partial x^\nu}(x^\mu)$.

We may write this as $\partial_\mu \mathcal{P}^\mu_i - \mathcal{L}_i \approx 0$, and where the momentum stress vector $\mathcal{P}^\mu_i$ is shorthand for $\mathcal{P}^\mu_i(u^j, u^j_\nu, x^\nu) := \frac{\partial \mathcal{L}}{\partial u^i_\mu}(u^j, u^j_\nu, x^\nu)$ and $\mathcal{L}_i$ is short hand for $\mathcal{L}_i(u^j(x^\nu), u^j_\nu(x^\mu), x^\nu) := \frac{\partial \mathcal{L}}{\partial u^i}(u^j(x^\mu), u^j_\nu(x^\mu), x^\nu)$.

The Euler-Lagrange equations are a set of N second order partial differential equations in the spacetime coordinates x^μ for the fields $u^i(x)$.

Energy momentum tensor

We may use the Euler-Lagrange equations to write $\partial_\mu \left(T^\mu_\nu(x^\mu) \right) \approx -\frac{\partial \mathcal{L}}{\partial x^\mu}$ where $T^\mu_\nu(x^\mu)$ is the canonical energy-momentum tensor density $T^\mu_\nu(x^\mu) := \mathcal{P}^\mu_i u^i_\nu - \delta^\mu_\nu \mathcal{L}$ [12], and $\frac{\partial \mathcal{L}}{\partial x^\mu}$ is the partial derivative of $\mathcal{L}(u^i, u^i_\mu, x^\mu)$ with respect to the explicit x^μ dependence of $\mathcal{L}$. We see that, if the Lagrangian density $\mathcal{L}$ is not explicitly a function of x^μ, and so has translation invariance, the energy momentum T^μ_ν is a conserved Noether current $\partial_\mu T^\mu_\nu(x^\mu) \approx 0$ and the total energy-momentum $P^0_\nu(t)$, which is $T^\mu_\nu(x^\mu)$ integrated over a spatial slice in Minkowski space, is a constant of motion (assuming $T(x) \to 0$ as x goes to spatial infinity.) P^0_ν transforms as a spacetime 1-form under Lorentz transformations, assuming $\mathcal{L}$ is a Lorentz density.

Infinitesimal variation of the action as a result of an arbitrary varying the field values and the spacetime coordinates is

$$\delta_\epsilon S[\phi] = \delta_\epsilon \int_B L(j^1\phi) \mathrm{d}^n x = \int_B E_i \delta_\epsilon u^i - \partial_\mu (T^\mu_\nu \delta_\epsilon x^\nu - \mathcal{P}^\mu_i \delta_\epsilon u^i) \mathrm{d}^n x \tag{2.36}$$

where $E_i(x) := -\partial_\mu \left(\frac{\partial \mathcal{L}}{\partial u^i_\mu}(u^j(x^\nu), u^j_\nu(x^\lambda), x^\nu) \right) + \frac{\partial \mathcal{L}}{\partial u^i}(u^j(x^\nu), u^j_\nu(x^\mu), x^\nu)$ are the Euler-Lagrange functions. If the action is constant under a particular continuous one parameter variation $\epsilon(x)$ on-shell, then $K^\mu = T^\mu_\nu \delta_\epsilon x^\nu - \mathcal{P}^\mu_i \delta_\epsilon u^i$ is a conserved current on-shell and $\partial_\mu K^\mu = E_i \delta_\epsilon u^i$ for an off-shell configuration and a variation which leaves the action unchanged for that configuration.

2.3.2 Hamiltonian field theory

Fields may be considered as mechanical systems with a copy of a set of dynamical variables u^a, $a = 1 \ldots N$, at each spatial point (x, y, z) of spacetime. Dynamical variables have been indexed by i in the previous sections above and we now consider this index to range over the spatial points as well - thus this index can now be replaced by continuous local coordinates x, y, z etc. as well as a discrete index a for N different fields. Sums over the index i in the sections above now become integrations over spatial sections as well as a sum over a. The phase space is now infinite dimensional, and each point in the phase space represents a particular state of a time slice (over all space) of the field, the field viewed as defined at every individual point of spacetime. The time extended configuration and phase spaces are now bundles over time where the fibers have infinite dimension. The base space of the bundle, representing time, is treated differently from the spatial geometry. The spatial geometry, even the notion of spatial locality, is not part of the Hamiltonian formalism but is set up separately by how the spatial degrees of freedom and summations are defined inside the Hamiltonian. This makes the Hamiltonian in field theory awkward and non-covariant. Nevertheless it is necessary for the canonical quantization of fields.

In the multiphase-space field theory presented in the next chapter, Ch.3, the equivalent of the Hamiltonian is a function of multiphase space whose coordinates are the field variables, the d multimomenta and the d spacetime coordinates. It is a spacetime density like the conventional Lagrangian density of a field and so locality and spacetime covariance are built in by the formalism, and the spaces involved are finite dimensional fiber bundles over spacetime.

2.4 Symmetries and constraints

2.4.1 Symmetry: Lagrangian mechanics

A Lagrangian system is called *symmetric* when the action remains unchanged when the path or trajectory is changed in some defined way: $\delta_\epsilon S[q^i(t)] = S[q^i(t) + \delta_\epsilon q^i(t)] = 0$. The change $\delta_\epsilon q^i(t)$ should be defined for any path or trajectory $q^i(t)$ (and generally will depend on the path). We will consider the set of symmetries ϵ to form a group G under composition of variations. The group may be discrete - such as time reversal and space inversion symmetries - but we will concern ourselves here with continuous symmetries forming Lie groups. The corresponding infinitesimal symmetries will form a Lie algebra. From the Lie algebra it is possible to reconstruct the part of the Lie group connected to the identity, therefore in studying symmetry is is often only necessary to deal with the Lie algebra and its (infinitesimal) action rather than the full group of symmetries. Often it is convenient to consider one dimensional subgroups of the Lie group of symmetries, i.e. a one parameter group of symmetries.

If the action is invariant under some continuous group of symmetries then it is worthwhile to distinguish between two different situations, (1) global symmetries and (2) local symmetries.

(1) If the symmetry is a continuous one parameter group of symmetries, then we first consider the case of a global symmetry where, for a particular symmetry variation, each path is mapped into another path. To this kind of symmetry is associated a velocity phase-space observable which is a constant of motion on trajectories, that is, paths which obey the Euler-Lagrange

equations of motion. This is Noether's theorem. Specifically, if the infinitesimal symmetry variation is $\delta_\epsilon q^i = \epsilon Q^i(q^j, t)$ and $\delta_\epsilon t = \epsilon T(q^j, t)$ with $\delta_\epsilon S[q^i(t)] = 0$, then, when the Euler-Lagrange equations are satisfied, the variation of the action integral (2.19) is

$$0 = \delta_\epsilon\ S[q^i] = \left[\ (\left(\frac{\partial L}{\partial v^j}\right) v^j - L)\,\delta_\epsilon t - \left(\frac{\partial L}{\partial v^j}\right)\delta_\epsilon q^j\ \right]_{\partial C(\mathfrak{T}Q)}$$

$$= \epsilon[\,(\left(\frac{\partial L}{\partial v^j}\right) v^j - L)\,T - \left(\frac{\partial L}{\partial v^j}\right) Q^j]_{t_f} - \epsilon[\,(\left(\frac{\partial L}{\partial v^j}\right) v^j - L)T - \left(\frac{\partial L}{\partial v^j}\right) Q^j]_{t_i} \tag{2.37}$$

showing that $[\,((\frac{\partial L}{\partial v^j})\,v^j - L)\,T - (\frac{\partial L}{\partial v^j})\,Q^j\,]$ has the same value at the end-points (initial time t_i and final time t_f) on the trajectory, where the E-L equations hold, C, and is therefore a constant on a trajectory (because the final end-point can be moved to any point on the trajectory). This situation often arises from physical symmetries such as space-translation symmetry, in which case the momentum is the corresponding constant of motion.

(2) A second type of symmetry is where the symmetry variation is local, in the sense that each path can be varied independently at each point in time (in the case of local field theories it may be the case that the field can be varied independently at each point in space and time), with the action remaining unchanged. Usually we are concerned with infinite dimensional *gauge groups* whose elements are time- (or, in local field theory, spacetime-) varying elements of a finite dimensional Lie group G, i.e. smooth functions from time or spacetime to G. In this case one can see that, keeping the endpoints fixed, the path can varied in between without changing the action. This is a situation where the action functional does not have a unique distinct stationary solution (trajectory), for given endpoints. In this case the Euler-Lagrange equations will be degenerate and the action principle is not sufficient to specify one particular trajectory, and thus there will therefore be a class of trajectories which satisfy the action principle for given fixed end points. One may view this as an incompletely specified mechanical system, or alternatively, a system where more degrees of freedom are used to specify the system than are strictly necessary. The extra non-physical degrees of freedom is the gauge freedom to vary observables and trajectories, without changing the gauge class of that trajectory, each class representing one physical trajectory. A well known example are covariant general relativistic actions where there is the gauge freedom of varying the underlying spacetime coordinate system - embodying a geometrical principle of general coordinate invariance. In these cases there are gauge degrees of freedom, the coordinate system, which are not physical, and there are the physical objects, such as the metric tensor and other fields, which have corresponding gauge variations (covariance). It may be simpler or more natural to write out the action with covariant components, but it must be in such a way that the covariance of the components 'cancel out' and leave the action invariant under the (non-physical) gauge variation. The action will then be dependent on the physical variations of the metric and the fields. As well as the action being simpler to write out, some features of the analysis of the dynamical system are easier to deal

with in this form, but for many other features such as the quantum functional integral, the physical degrees of freedom have to be correctly isolated from the gauge degrees of freedom.

There are several interrelated methods of dealing with such gauge theories. One is to solve for the gauge degrees of freedom explicitly from the physical degrees and constructing the physical Lagrangian by eliminating the non-physical degrees of freedom. However it may not be possible to solve explicitly for the gauge degrees of freedom. The gauge freedom is usually the result of a simpler or more natural description of the system, so removing it will usually make for a more complicated or intractable description, thereby removing features such as symmetries, locality, etc, useful for finding solutions, renormalization etc. A second method is 'gauge fixing', specifying the values of the gauge variables so as to pick out one representative from each gauge equivalence class of trajectories. This often involves adding a gauge fixing term to the Lagrangian so that the system is no longer gauge invariant and thus the action principle picks out a particular value of the gauge variables. A generalization of this is the addition of a *gauge breaking* term where the gauge is fixed by the action principle (e.g R_ξ gauges). This latter does not remove the gauge degrees of freedom, but if the result is independent of the choice of gauge, the physical degrees of freedom have been decoupled from the gauge degrees of freedom. A gauge fixing term in a Feynman path integral is often employed, which has the advantage of providing a well defined Green's function required for pertubation theory. It is necessary to ensure that the result is independent of the choice of gauge fixing: the Fade'ev-Popov ghost fields were invented to take care of this [64], because the path integral of the ghosts designed to be equal to the required Jacobian factor. The disadvantage of gauge fixing is the appearance of negative norm states and the loss of the gauge symmetry useful for renormalization. Another possibility for certain systems is that a global section of the gauge bundle may not exist (the Gribov problem, see 4.1). It should be said that that the gauge invariant fields may not be fundamental: as the Aharonov-Bohm effect shows, the gauge variant potential is more physically fundamental. A third method is a kind of combination of the above which is the BRST approach, described in chapter 4. Here the dynamical system is extended with extra opposite parity degrees of freedom ('ghosts') corresponding to the gauge variation parameters, and then one modifies the Lagrangian with a gauge fixing term and extra terms so that the gauge symmetry is replaced by the global BRST supersymmetry. This combines the advantages of both methods: gauge fixing to a specific trajectory and retaining symmetry, both particularly useful in quantum field theory in particular. BRST is an homological algebraic approach which meshes well with the Poisson algebra of observables in classical and quantum mechanics and will be described below in both the phase-space and multiphase-space versions.

The abstract relationship between the original phase space with symmetry and the physical reduced phase space in terms of symplectic manifolds is known as Marsden-Weinstein reduction [6] described in the next section (2.4.2).

2.4.2 Hamiltonian constraints and Marsden-Weinstein symplectic reduction

Lie group action on a symplectic manifold

A summary is contained in [53] and in [61] and a more extensive treatment in the book by Marsden and Ratiu [92]. We will consider a certain class of symmetries: diffeomorphisms ϕ_g of a symplectic manifold, which are also symplectomorphisms (that is, that the symplectic form is preserved: $\phi_g^*\Omega = \Omega$). We assume a system where the symmetries are the left-action by elements g of a Lie group G, on the symplectic manifold $\mathbb{M}$. We assume that the symplectomorphisms are also hamiltonian - that is, the infinitesimal elements around the identity element I of G are infinitesimal flows on $\mathbb{M}$ which are hamiltonian vector fields. We also assume that the symplectomorphisms are Poisson (defined below) as well as hamiltonian - that is, the Lie algebra of the infinitesimal generators of G map to the Poisson algebra of the hamiltonian functions corresponding to the infinitesimal flows on $\mathbb{M}$ of a hamiltonian action. It will be shown that a Poisson group action, if free and proper, will lead to a *constraint submanifold* $\mathbb{M}_G \subset \mathbb{M}$, which is a presymplectic submanifold of $\mathbb{M}$. This submanifold is in turn foliated by the orbits of the symmetry transformation. Under the above conditions and the conditions that the constraint is regular, the space of orbits (displaying various notations seen in the literature), $\mathbb{M}_G/G = (\mathbb{M}_G)^G =: \mathbb{M}//G =: \tilde{\mathbb{M}} =: \mathbb{M}_{GG}$, is a symplectic manifold with a symplectic form inherited, via the embedding, from the original symplectic manifold, and is denoted the reduced phase space (theorem 1 in [6]). In many applications, this reduced phase space is often the physical phase space, whereas the original phase space has non-physical gauge or symmetry degrees of freedom, which it may be necessary or useful to retain for part of the study of the system, but from which we finally want to obtain the dynamics on the reduced phase space. This is what is known as the Marsden-Weinstein reduction [6]. If the dynamical Hamiltonian H on the original phase space M is invariant under the action of the symmetry group G: $H(g \cdot x) = H(x)\ \forall g \in G$, then H is well defined on the reduced phase space $\mathbb{M}//G$.

Hamiltonian action

First the notion of hamiltonian symplectomorphism is defined.

The equation $X_F \lrcorner \Omega = \mathrm{d}F$, where F is any function (0-form) on a symplectic manifold $(\mathbb{M}, \Omega)$, has a unique solution $X_F = \Omega^{-1} \lrcorner \mathrm{d}F = \{\cdot, F\}$ (Ω^{-1} exists as a map from 1-forms to vectors because Ω is non-degenerate), which is a vector field on $\mathbb{M}$, denoted the hamiltonian vector field generated by F. This vector field, as an infinitesimal transformation of the manifold, is an infinitesimal symplectomorphism of $(\mathbb{M}, \Omega)$ because: $\mathcal{L}_{X_F}\Omega = X_F \lrcorner \mathrm{d}\Omega + \mathrm{d}(X_F \lrcorner \Omega) = X_F \lrcorner 0 + \mathrm{dd}F = 0$ (because $\mathrm{d}\Omega = 0$ and $\mathrm{dd} = 0$). Such a symplectomorphism is called a hamiltonian symplectomorphism. Conversely, any vector field X_F which satisfies the equation $X_F \lrcorner \Omega = \mathrm{d}F$ for some F is called a hamiltonian vector field and is a generator of a one dimensional Lie group of global hamiltonian symplectomorphisms of $\mathbb{M}$, the flow of the vector field X_F, and F is called the hamiltonian function or observable corresponding to that hamiltonian vector field. If the one-form $X_\xi \lrcorner \Omega = \xi$ is closed rather than exact, the vector field X_ξ is called a locally hamiltonian vector field and is a generator of a one dimensional Lie group of local symplectomorphisms of $\mathbb{M}$. ξ is called the hamiltonian form corresponding to that locally hamiltonian vector field. On a simply connected open patch a locally hamiltonian vector field is hamiltonian. If $\mathfrak{g}$ is a Lie algebra acting on a symplectic manifold by such infinitesimal hamiltonian symplectomorphisms, then there is a hamiltonian vectorfield $\eta_{\mathbb{M}}$ and observable (called a constraint in this context) $h_{\eta_{\mathbb{M}}}$ corresponding to each element $\eta \in \mathfrak{g}$. Such an action is called a hamiltonian action of the Lie algebra and there is a hamiltonian map from the Lie algebra to the space of functions on $\mathbb{M}$.

Poisson action

If there is a hamiltonian action of the Lie algebra $\mathfrak{g}$ then, up to a certain cohomological obstruction (if $h_{[\eta^1_{\mathbb{M}}, \eta^2_{\mathbb{M}}]} - \{h_{\eta^1_{\mathbb{M}}}, h_{\eta^1_{\mathbb{M}}}\} = \mathrm{d}\theta(\eta^1_{\mathbb{M}}, \eta^2_{\mathbb{M}})$, for some 1-form θ on $\mathbb{M}$), there exists a hamiltonian map $\tilde{h}$ such that there is a Lie algebra homomorphism $\tilde{h} : \mathfrak{g} \longrightarrow (C^\infty(\mathbb{M}), \{\})$ from the Lie algebra $\mathfrak{g}$ of the symmetry group to the Poisson algebra of functions on the symplectic manifold. Such constraints h_η form a sub-algebra, the constriant algebra, of the Poisson algebra $(C^\infty(\mathbb{M}), \{\})$, the algebra of classical observables. This is called a first class set of constraints in the Dirac terminology, which are constraints which form a Poisson algebra, which is the case here. Such an action is called a Poisson action of a Lie algebra on a symplectic manifold.

Moment map

The transpose of the above map, from the Lie algebra to the Poisson algebra of the constraints, is the map from the symplectic manifold to the Lie algebra dual, and is called the moment map: $\mathfrak{J} : \mathbb{M} \longrightarrow \mathfrak{g}^* :: m \mapsto v = \mathfrak{J}(m)$, where $\mathfrak{J}$ is defined by $h_{\eta_{\mathbb{M}}}(m) = \langle \mathfrak{J}(m), \eta \rangle$, where $\langle v, \eta \rangle$ indicates the dual pairing of $v \in \mathfrak{g}^*$ and $\eta \in \mathfrak{g}$. If there is a Lie algebra homomorphism as above then the moment map intertwines between the canonical coadjoint action of G on $\mathfrak{g}^*$ and the original Lie group action on $\mathbb{M}$ (this is called G-equivariance).

Symmetry of the Hamiltonian

The Hamiltonian H is said to be symmetric under the action of the Lie group G when the following holds: the dynamical Hamiltonian function H of a mechanical system, which generates the time evolution (i.e. $X_H = -\{H, \cdot\}$, is the instantaneous rate of change in phase space), is invariant ($X_a \lrcorner \mathrm{d}H = -\{h_a, H\} = 0$) under the Lie group G, with Lie algebra hamiltonian action $X_a = \rho(\eta_a)$, where the η_a are a basis of the Lie algebra and ρ is the action (infinitesimal diffeomorphism) on the manifold $\mathbb{M}$. Then, corresponding to the hamiltonian vector fields $X_a = \{\cdot, h_{X_a}\}$, there will be a hamiltonian functions h_{X_a} with $X_a \lrcorner \Omega = \mathrm{d}h_{X_a}$ and,

$$0 = X_a(H) = X_a \lrcorner \mathrm{d}H = X_a \lrcorner X_H \lrcorner \Omega = -X_H \lrcorner \mathrm{d}h_{X_a} = \{h_{X_a}, H\} \approx \frac{dh_{X_a}}{dt}. \tag{2.38}$$

Thus the observables h_{X_a} corresponding to the symmetries X_a of the Hamiltonian are constants of motion. This is an expression of Noether's theorem - that there is a constant of motion assoaciated with every one-dimensional Lie group of symmetries.

The symmetry is said to commute with the Hamiltonian.

Marsden-Weinstein reduction

One may choose a particular value of h: $h_{X_a} = 0$, for all $a = 1 \ldots K$, and consider the locus of points $\mathbb{M}_G \subset \mathbb{M}$ which are solutions of this constraint equation, the zero level set of the h's in $\mathbb{M}$. $\mathbb{M}_G$ is the same as $\mathfrak{J}^{-1}(\bar{0})$, the kernel of the moment map. If $\bar{0}$ is a regular value of h, i.e. the Jacobean matrix of h has constant rank K on the zero locus, then $\mathbb{M}_G$ is a coisotropic submanifold of $\mathbb{M}$. [A submanifold of a symplectic manifold is called *coisotropic* if the kernel $\mathfrak{T}\mathbb{M}_c^{\perp}$ of Ω_c, (i.e. $Y \in \mathfrak{T}\mathbb{M}_c^{\perp}$ iff $Y \lrcorner \Omega \llcorner X = 0 \ \ \forall X \in \mathfrak{T}\mathbb{M}_c$), where Ω_c is the form Ω restricted to $\mathfrak{T}\mathbb{M}_c$, where $\mathbb{M}_c$ is a submanifold of $(\mathbb{M}, \Omega)$, lies entirely inside $\mathfrak{T}\mathbb{M}_c$: $\mathfrak{T}\mathbb{M}_c^{\perp} \subset \mathfrak{T}\mathbb{M}_c$.] This is because $0 = Y \lrcorner \Omega \llcorner X_h = Y \lrcorner \mathrm{d}h$ implies that Y lies in the $h =$ constant hypersurface, in this case the zero locus. If the dimension of the kernel in a coisotropic submanifold $\mathbb{M}_c$ is constant on $\mathbb{M}_c$, then the tangent spaces $\mathfrak{T}_m\mathbb{M}_c^{\perp} \subset \mathfrak{T}_m\mathbb{M}_c$ ranging over $m \in \mathbb{M}_c$ define a distribution (a subspace of the tangent space at each m, i.e. a subbundle of the tangent bundle) called the characteristic distribution of Ω_c, and this distribution is Frobenius integrable (defined below), and as a consequence the coisotropic submanifold foliates into connected submanifolds whose tangent spaces are this distribution.

The Frobenius condition is that the Lie bracket of vector fields lying in the distribution is in the distribution, $[X_a, X_b] \in \mathfrak{T}\mathbb{M}_c^{\perp}, \forall$ local sections $X_a, X_b \in \mathfrak{T}\mathbb{M}_c^{\perp}$, and is a consequence of the closure, $\mathrm{d}\Omega_c = 0$, (which results from the closure of Ω and the commuting of the exterior derivative d with the embedding map,) of the presymplectic form Ω_c: If X_a, X_b are any local sections in $\mathfrak{T}\mathbb{M}_c^\perp$ and Y is any local section in $\mathfrak{T}\mathbb{M}_c$ then,

$$0 = (\mathrm{d}\Omega_c)(Y, X_a, X_b)$$

$$= Y(\Omega_c(X_a, X_b)) + X_a(\Omega_c(X_b, Y)) + X_b(\Omega_c(Y, X_a)) - \Omega_c([Y, X_a], X_b) - \Omega_c([X_a, X_b], Y)$$

$$-\Omega_c([X_b, Y], X_a) \;=\; -\Omega_c([X_a, X_b], Y) \tag{2.39}$$

Thus $\Omega_c([X_a, X_b], Y) = 0$ for all such $Y \in \mathfrak{T}\mathbb{M}_c$, so $[X_a, X_b] \in \mathfrak{T}\mathbb{M}_c^\perp$. Because the X_a's are a basis of the Lie algebra action, $[X_a, X_b] = f_{ab}^k X_k$, the X_a's are involutive and thus can be integrated to foliate $\mathbb{M}_G$.

If the action of G is free and proper, the space of leaves of the G-orbits $\mathbb{M}_{GG} := \mathbb{M}//G := \widetilde{\mathbb{M}} := \mathbb{M}_G/G$ is a manifold as indicated above, and in fact this manifold has a well defined symplectic form $\Omega|_{\mathbb{M}//G}$. (If a Poisson action is not free and proper, then it will be locally free and the space of leaves will be an orbifold.) $\Omega|_{\mathbb{M}//G}$ is a symplectic form on the reduced space $\mathbb{M}//G$ because the presymplectic form $\Omega|_{\mathbb{M}/G}$ on $\mathbb{M}_G$, which is the symplectic form Ω in $\mathbb{M}$ pulled back to $\mathbb{M}_g$ by the embedding map (and which preserves the closure property of Ω), has as kernel precisely the vectors tangent to the leaves of the foliation. So the null directions to the presymplectic form are mod-ed out in $\mathbb{M}_G/G$, and the space of leaves thereby becomes a symplectic manifold, the symplectic quotient of $\mathbb{M}$ by G also known as 'the reduced phase space', $(\mathbb{M}//G, \Omega|_{\mathbb{M}//G})$. This process is known as Marsden-Weinstein reduction [6].

Example: exact symplectomorphism

Any infinitesimal symplectomorphism X on manifold with an exact symplectic form $\Omega = -\mathrm{d}\Theta$ is Poisson, because the function $h_X = X \lrcorner \Theta$ is Hamiltonian for X, and $h_{[X,Y]} = [X, Y] \lrcorner \Theta = X \lrcorner \mathrm{d}(Y \lrcorner \Theta) = X \lrcorner \mathrm{d}h_Y = \{h_X, h_Y\}$, where X and Y are infinitesimal symplectomorphisms.

Example: G action on Q

An example of a Poisson action is any Lie group diffeomorphic action G on a manifold Q naturally and equivariantly extended to the dual tangent bundle $\pi : \mathfrak{T}^*Q \longrightarrow Q$. The reduced phase space is $(\mathfrak{T}^*Q)//G = \mathfrak{T}^*(Q/G)$, if the action is free and proper on Q.

Example: the dual of a Lie algebra

The dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ of a Lie group G has a canonical foliation into symplectic manifolds, where the symplectic form is defined by

$$\eta_1 \lrcorner \eta_2 \lrcorner \Omega_{\mathfrak{g}}(v) \ := \ \langle v, [\eta_1, \eta_2]_{\mathfrak{g}} \rangle \ = \ \mathrm{d}v(\eta_1, \eta_2) \tag{2.40}$$

where the exterior derivative is on the manifold G and $v \in \mathfrak{g}^*$ can be viewed as a left-invariant one form on G, $\eta_1, \eta_2 \in \mathfrak{T}\mathfrak{g}^* \ \simeq \mathfrak{g} \simeq V^L(G)$, the space of left invariant vector fields on G. The left invariant vector fields η_1, η_2 can be viewed as the infinitesimal action of η_1, η_2 on G. $[\cdot,\cdot]_{\mathfrak{g}}$ is equivalently the bracket of the Lie algebra $\mathfrak{g}$ or the Lie bracket of vector fields on G. The moment map is the identity $v = \mathfrak{J}(v)$.

An example of a coisotropic manifold is the dual $\mathfrak{g}^*$ of a Lie algebra $\mathfrak{g}$ of a Lie group G, with the canonical presymplectic form given by $X \lrcorner Y \lrcorner \Omega = \langle \xi, [X, Y] \rangle$ with $X, Y \in \mathfrak{g}$ and $\xi \in \mathfrak{g}^*$. The coadjoint action $G \cdot \mathfrak{g}^*$, which is defined by $\langle g \cdot v, g \cdot X \rangle \equiv \langle v, X \rangle$ with $g \in G$, is a Poisson action where the moment map on $\mathbb{M} = \mathfrak{g}^*$ is the identity. The dual Lie algebra is foliated into coadjoint orbits $G \cdot \xi$, where $\Omega|_{G\cdot\xi}$ is a G-invariant symplectic form. This is the symplectic reduction of $\mathfrak{T}^*G$ with the canonical one form and symplectic form, where the action is the action of left multiplication by G naturally and equivariantly extended to $\pi : \mathfrak{T}^*G \longrightarrow G$. The moment map at any point (g, ξ) in T^*G is $\mathfrak{J}(g, \xi) = R_g^*\xi$, the pull back of ξ by right multiplication by g. The level set $\mathfrak{J}^{-1}(0)$ is a submanifold diffeomorphic to G. If $G_\xi \subset G$ is the coadjoint action stabilizer of ξ, then G_ξ acts by left multiplication on $\mathfrak{J}^{-1}(0)$ and the symplectic reduction is $\mathfrak{T}^*G//G_\xi = \mathfrak{J}^{-1}/G_\xi = G \cdot \xi$

In practice the original phase space will usually be simpler that the reduced phase space, where the latter is usually the desired physical system, and the former the space in which the dynamical system is originally defined. There are several routes to the reduced phase space: one is to eliminate the non-physical degrees of freedom directly at the beginning, another are the Dirac method of constraints, and the BRST homological approaches are yet another. Dirac cohomology which is related to BRST is yet another [78]. Often, for the reason indicated above and also to retain the manifest symmetry of the system as much as possible, especially in quantization, it is desirable to work with the original phase space rather than the reduced phase space. This is what the BRST methods, described below, achieve, as well as providing other advantages. The BV formalism is a well known method of constructing a BRST model. The BRST method is constructed from the Poisson algebra of observables on phase space rather than phase space itself as described above. Marsden-Weinstein reduction expressed using observables is described in section 4.2.1.

Example of symmetry and constraints: The electromagnetic field.

The analysis of the electromagnetic field on Minkowski space which illustrates the ideas in this chapter is to be found in Appendix C: 'The electromagnetic field in phase space'.

2.5 Summary

In this chapter classical Hamiltonian and Lagrangian mechanics was reviewed as a prologue to the covariant Hamiltonian mechanics in the next chapter. The concepts in this chapter we employed in the analysis of the electromagnetic field in the appendix. The section on symmetry and the example of the electromagnetic field will be employed in the chapter on BRST. The review of Hamilton-Jacobi theory and canonical transformations has been relegated to the appendix.

Chapter 3

Multisymplectic field theory

The multisymplectic formalism for fields will be presented in this chapter as was symplectic mechanics in the previous chapter, by generalizing the analogous notions. The Yang-Mills field in the multiphase space formalism is described in detail as an example of the multisymplectic formalism in this chapter (and can be compared to the phase space formulation in the appendix) and the results of this are employed in the example of the BRST construction in chapter 4. Several other less detailed examples are given in the appendix. The use of multivectors to characterise field solutions is relegated to the appendix and the both the phase space and multiphase space Hamilton-Jacobi theory are also in the appendix.

3.1 Multiphase space

This section is based on [68] [28] [54]. Here the analogue of extended phase space will be the multiphase space $\mathcal{M}$, and the analogue of phase space will be the covariant multiphase space $\mathcal{P}$. Both these manifolds have a multisymplectic form, which plays a role similar to the symplectic form in Hamiltonian mechanics.

The Hamiltonian mechanics formalism in the last chapter starts from a bundle, $Q \times \mathbb{R}$, with fiber Q and base space $\mathbb{R}$ (for time t), which is the natural extended configuration space for a mechanical system, whose sections would be parametrized paths $q^i = q^i(t)$ in configuration space. Analogously, the multisymplectic formalism begins with the field configuration bundle $\mathcal{E}$ over an d-dimensional space-time manifold B (where a point x is specified in a coordinate patch by coordinates $(x^\mu) := (x^0, \ldots x^{d-1})$, whose sections would be specific values of the fields, $u^i = u^i(x^\mu)$, $i = 1 \ldots N$, defined over spacetime B. The fiber $\mathbb{U}$ at any point in spacetime is the target space of possible field values at that point. In the study of fundamental physical fields, the fiber is often a vector space, as in Yang-Mills theories, or a manifold with extra structure as in sigma models. In this chapter we assume that it is $\mathbb{R}^N$.

The analogue of the velocity phase space $\mathfrak{T}Q$ with local coordinates q^i, v^i, is the *first jet bundle* (for first order field theories) which is the bundle of the first derivatives of sections of $\mathcal{E}$:

$$\pi^{\mathcal{E},J^1\mathcal{E}} : J^1E \longrightarrow \mathcal{E} :: (x^\mu, u^i, u^i_\mu) \mapsto (x^\mu, u^i) \tag{3.1}$$

of the field configuration bundle, $\pi^{B,\mathcal{E}} : \mathcal{E} \longrightarrow B :: (x^\mu, u^i) \mapsto (x^\mu)$, where B is the underlying d-dimensional spacetime manifold. A spacetime field configuration is given by a section $u : B \longrightarrow \mathcal{E} :: x \mapsto (u^i)(x)$, of the field configuration bundle $\pi^{B,\mathcal{E}}$, which has a prolongation to the following section of $\pi^{B,J^1\mathcal{E}}$:

$$j^1u : B \longrightarrow J^1\mathcal{E} :: (x) \longmapsto (x^\mu, u^i, u^i_\mu) = j^1u(x) = (x^\mu, u^i(x), \frac{\partial u^i(x)}{\partial x^\mu}) \tag{3.2}$$

The analogue of the Lagrangian is the Lagrangian density d-form, which for a specified system is a defined bundle map over B, $\mathbb{L} : J^1\mathcal{E} \longrightarrow \Lambda^d B$, where $\Lambda^d B$ is the bundle of d-forms on spacetime B. The Lagrangian density d-form is $\mathbb{L}(x^\mu, u^i, u^i_\mu) = \mathcal{L}(x^\mu, u^i, u^i_\mu)\mathrm{d}^d x$, where $\mathbb{L}$ is a spacetime density form on the total space of the first jet bundle. The action, $S[u^i(x)]$, of the field configuration $u^i(x)$ in the region $B' \subset B$ of spacetime B is equal to the integral of the Lagrangian density d-form over the region B':

$$S[u^i(x)] = \int_{B'} \mathbb{L}(j^1u) = \int_{B'} \mathcal{L}(x^\mu, u^i, u^i_\mu)\mathrm{d}^d x \tag{3.3}$$

This is simply the conventional Lagrangian density formulation of a field theory as in section 2.3.1. But now we will introduce multimomenta.

The analogue of the extended phase space, denoted the *multiphase space*, $\mathcal{M}$, is the space of affine bundle maps from $J^1\mathcal{E}$ to $\mathcal{E} \times \Lambda^d B$, and has local coordinates (x^μ, u^i, p^μ_i, p). An affine map being, in local coordinates, $(p^\mu_i, p) :\ (x^\mu, u^i, u^i_\mu) \longmapsto (x^\mu, u^i, p^\mu_i u^i_\mu + p)$. This is 'affine' rather than 'linear' because of the p parameter of the map. The coordinates p^μ_i are called *multimomenta* (the $d-1$-form $p^\mu_i \mathrm{d}x_\mu$ is 'canonically dual' to u^i) and p is the 'energy density' coordinate. The multiphase space is canonically isomorphic to a certain subbundle of the bundle of d-forms on $\mathcal{E}$: $\Lambda^d_1\mathcal{E} \subset \Lambda^d\mathcal{E}$, namely the bundle of $(d-1)$-horizontal d-forms ω - which give zero when contracted with any two vertical vectors v, w in $V\mathcal{E} \subset \mathfrak{T}\mathcal{E}$:

$$\begin{aligned}\Lambda^d_1\mathcal{E} &= \{\omega \in \Lambda^d\mathcal{E} \mid\ v \lrcorner\ w \lrcorner\ \omega = 0\ \ \forall v, w \in V\mathcal{E}(\text{ i.e. } \pi^{B,\mathcal{E}}_* v = 0 = \pi^{B,\mathcal{E}}_* w)\} \\ &= \{(\mathrm{d}u^i \wedge\ p^\alpha_i \mathrm{d}^{d-1}x_\alpha + p\ \mathrm{d}^d x)\}\end{aligned} \tag{3.4}$$

in local coordinates.

On any bundle of forms on a manifold one can define the canonical tautological form on the total space. Here, because of the above definition of the multiphase space $\mathcal{M} \simeq \Lambda_1^d\mathcal{E}$ as a bundle of forms, one can define a canonical tautological d-form and corresponding canonical $d+1$-form, locally,

$$\widetilde{\Theta} = \mathrm{d}u^i \wedge\ p_i^\alpha \mathrm{d}^{d-1}x_\alpha + p\ \mathrm{d}^d x \tag{3.5}$$

$$\widetilde{\Omega} := -\mathrm{d}\widetilde{\Theta} = -(\mathrm{d}p_i^\alpha \wedge \mathrm{d}u^i \wedge \mathrm{d}^{d-1}x_\alpha + \mathrm{d}p \wedge \mathrm{d}^d x) \tag{3.6}$$

These forms are defined on the manifold $\Lambda_1^d\mathcal{E}$. The canonical $d+1$ form is a multisymplectic form, meaning that it is closed, $\mathrm{d}\widetilde{\Omega} = 0$, and 1-non-degenerate, i.e. $X \lrcorner \widetilde{\Omega} = 0 \implies X = 0\ \forall X \in \mathfrak{T}M$ (this is not generally true when X is a higher degree multivector). In the case $d = 1$, when we simply consider spacetime $B = \mathbb{R}$ to be the time axis, the extended multiphase space reduces to extended phase space and the multisymplectic canonical forms reduce to the symplectic canonical forms $\widetilde{\Theta}$ and $\widetilde{\Omega}$ on the extended phase space $T\widetilde{Q}^*$ as described in the symplectic mechanics presented in the previous chapter.

The analogue of the phase space (the dual tangent bundle, $\pi : \mathfrak{T}^*Q \longrightarrow Q$, with local coordinates q^i, p_i, to the configuration space Q), is the dual of the first jet bundle, $\pi^{\mathcal{E},J^1\mathcal{E}^*} : J^1\mathcal{E}^* \longrightarrow \mathcal{E}$, called the *covariant phase space* or *covariant multiphase space* $\mathcal{P}$, which has local coordinates (x^μ, u^i, p_i^μ). This is canonically isomorphic to a bundle quotient of certain bundles of d-forms on $\mathcal{E}$: $J^1\mathcal{E}^* \simeq \frac{\Lambda_1^d}{\Lambda_0^d} = \{(\mathrm{d}u^i \wedge\ p_i^\alpha \mathrm{d}^{d-1}x_\alpha)\}$, where Λ_1^d is defined in the previous paragraph and Λ_0^d is the space of d-horizontal d-forms on $\mathcal{E}$: $\Lambda_0^d = \{\omega \in \Lambda^d\mathcal{E} \mid v \lrcorner \omega = 0\ \forall v$ such that $\pi_*^{B,\mathcal{E}} v = 0\} = \{(p\mathrm{d}^d x)\}$.

We will also use the phrase 'multiphase space' loosely to refer to a covariant multiphase space. The specific definitions will indicate when this is the case.

There is locally a canonical tautological d-form on the covariant multiphase space: $\Theta = p_i^\mu \mathrm{d}u^i \wedge \mathrm{d}x_\mu$ locally, and canonical $(d+1)$-form: $\Omega := -\mathrm{d}\Theta = \mathrm{d}u^i \wedge \mathrm{d}p_i^\mu \wedge \mathrm{d}x_\mu$ locally.

DeDonder-Weyl Hamiltonian

The analogue of the dynamical Hamiltonian function on the phase space is a spacetime horizontal d-form $\mathcal{H}\mathrm{d}^d x$ on the dual of the jet bundle: $\mathcal{H} : J^1\mathcal{E}^* \longrightarrow \Lambda^d(B) :: (x^\mu, u^i, p_i^\mu) \longmapsto \mathcal{H}(x^\mu, u^i, p_i^\mu)\,\mathrm{d}^d x$ locally or, more generally, a section of the bundle $\pi^{J^1\mathcal{E}^*,\mathcal{M}} : \Lambda_1^d \longrightarrow \frac{\Lambda_1^d}{\Lambda_0^d}$. In a local trivialization patch, $p\,\mathrm{d}^d x = \mathcal{H}(x^\mu, u^i, p_i^\mu)\,\mathrm{d}^d x$. The function $\mathcal{H}$ is called the *DeDonder-Weyl Hamiltonian*.

Equation of motion

The natural generalization of the symplectic equation of motion (2.2) is the multisymplectic 'equation of motion',

$$X_F \lrcorner \Omega = \mathrm{d}F, \tag{3.7}$$

where Ω is a $d+1$ multisymplectic form, and, in the case of homogenous form degree, $F \in \Lambda^p \mathfrak{T}^* \mathcal{M}$ is a p-form and $X_F \in \Lambda^{d-p} \mathfrak{T} \mathcal{M}$ an $d-p$ multivector, will in general have many or no solutions for given F or X_F and so the situation is more complicated than in symplectic case where $d = 1$, even for vector fields [67]. Given the particular construction of the canonical multisymplectic form (3.6) on the multiphase space, there are strong constraints on F for there to exist a multivector field X_F which satisfies (3.7) (examined in [38]). Such p-forms are called hamiltonian p-forms.

Of particular interest are forms of degree: (1) $p = 0$,

In this case, F is a hamiltonian 0-form (function) with corresponding hamiltonian d-multivectorfield. For instance, the function F may be the analogue of the dynamical Hamiltonian function, the DeDonder Weyl 'Hamiltonian' (- see below in section 3.4) and the d-multivectorfields X_F define the tangent plane to the d-dimensional spacetime hypersurfaces which are particular field solutions where (3.7) are the Hamilton's equations of motion (this is examined in appendix E).

Also of particular interest are forms of degree: (2) $p = d - 1$.

In this case, F is a hamiltonian $(d-1)$-form with corresponding unique (because of the 1-non-degeneracy of the multisymplectic form) hamiltonian vectorfield, which can represent, for instance, symmetry transformations or a derivation on the algebra of observables on multiphase space. A particular class of infinitesimal diffeomorphisms on multiphase space are infinitesimal multisymplectomorphisms. A multisymplectomorphism is a diffeomorphism Φ on a multisymplectic manifold $\mathcal{M}$ which preserves the multisymplectic form: $\Phi^*\Omega = \Omega$.

There have been several approaches to dealing with the degeneracy of equation (3.7). One can consider equivalence classes of hamiltonian p-forms mod closed forms ($\mathrm{d}F = 0$) and $d - p$ multivectorfields mod characteristic mutivectorfields ($X \lrcorner \Omega = 0$). In order to obtain a (graded) Posson structure, the authors in [46], [47], [93], [83] generalize the notion of multivectors to certain form valued multivectors, called generalized hamiltonian multivectorfields, $\check{X}_{F\wedge G} = (-1)^{(p+1)q} G \wedge X_F + (-1)^p F \wedge X_G$ where p and q are the form degrees of the hamiltonian forms F and G respectively, and X_F and X_G are their corresponding (not unique) $d-p$ and $d-q$ multivectorfields. Then $\check{X}_{F\wedge G} \lrcorner \Omega = \mathrm{d}(F \wedge G)$. Such a $F \wedge G$ is not generally a hamiltonian form, so this definition of 'generalized hamiltonian multivectorfields' allows the wedge products of hamiltonian forms to be 'generalized hamiltonian forms'. This would be the analogue of the ring, under pointwise multiplication and addition, of observables in symplectic mechanics.

In the special case where F is a hamiltonian $(d-1)$-form with a corresponding (unique) hamiltonian vectorfield X_F, the Lie derivative of Ω with respect to X_F is zero:

$$\mathcal{L}_{X_F}\Omega = X_F \lrcorner \,\mathrm{d}\Omega + \mathrm{d}(X_F \lrcorner\, \Omega) = X_F \lrcorner\, 0 + \mathrm{dd}F = 0, \tag{3.8}$$

where $\mathcal{L}_{X_H}$ is the Lie derivative of flow of the vector field X_F. This is the same as for a hamiltonian symplectomorphism (2.7). Thus X_F is an infinitesimal *hamiltonian multisymplectomorphism* generated by F. If $\mathcal{L}_X\Omega = 0$ for some vector field X then X is a multisymplectomorphism which is locally hamiltonian. We define $\mathcal{H}^{d-1}(\mathcal{M},\Omega)$ to be the set of hamiltonian $d-1$-forms, $\mathrm{Ham}^1(\mathcal{M},\Omega)$ to be the additive space over $\mathbb{R}$ of hamiltonian vectorfields. Note that if another $d-1$-form F' is closed ($\mathrm{d}F' = 0$), then $F + F'$ has then same unique hamiltonian vectorfields as F. This is the generalization of the fact that observables which differ by a constant generate the same vector fields on phase space. Because Ω is 1-non-degenerate, then $\mathcal{H}^{d-1}(\mathcal{M},\Omega)$ modulo closed $d-1$-forms, $\tilde{\mathcal{H}}^{d-1}(\mathcal{M},\Omega)$, is isomorphic to $\mathrm{Ham}^1(\mathcal{M},\Omega)$: $\tilde{\mathcal{H}}^{d-1}(\mathcal{M},\Omega) \simeq \mathrm{Ham}^1(\mathcal{M},\Omega)$

3.2 Multi-Poisson brackets

The Poisson algebra of functions (observables) on phase space is a powerful tool in classical mechanics and is the starting point for canonical quantization where a mapping of the Poisson algebra to a Lie algebra on a Hilbert space is sought. Is there a comparable structure for observables on multiphase space? In this subsection an obvious generalization of the Poisson brackets is studied. This object is used in this chapter, the chapter on multisymplectic BRST, and in the chapters on the topological sigma model. Other brackets are examined in the appendix.

In the multisymplectic setting, we have observables like $f(u^i)p_i^\mu \mathrm{d}^{d-1}x_\mu$ and $f(u^i)p_i^\mu \mathrm{d}^{d-1}x_\mu \wedge *p^i_\mu \mathrm{d}x^\mu$, which are forms rather than functions on which the brackets would act. In keeping with our approach to keep the most tractable formalism for use in the examples later, rather than generality, observables are usually covariant 0, $d-1$ or d-forms built up from objects like $u^i, p_i^\mu \mathrm{d}^{d-1}x_\mu, *p_{\mu i}\mathrm{d}x^\mu$. Usually we will have observables linear or quadratic in the multimomenta.

We are seeking to keep certain properties, in particular the Jacobi identity and the derivation property on products, for at least generators of symmetry transformations. Starting from the multisymplectic $d+1$ form, we note that hamiltonian p-forms have corresponding $(d-p)$

hamiltonian multivector fields. Certain form degrees are of particular interest: 0 or d-forms such as the DDW (DeDonder-Weyl) Hamiltonian, and the field observables u^i, which has a corresponding d-degree Hamiltonian multivector field. In the case of the DDW Hamiltonian, we expect the d-degree Hamiltonian multivector field to represent the tangent to the trajectory in multiphase space. We are also interested in $(d-1)$-forms such as the multimomentum $p_i^\mu \mathrm{d}^{d-1}x_\mu$ which has a corresponding 1-degree Hamiltonian vector field. This case of interest for infinitesimal symmetry algebras where the symmetry acts by a vector field on multiphase space. We are also interested in 1-forms such as the $u^i_{,\mu}\mathrm{d}x^\mu$ and $*p_{\mu i}\mathrm{d}x^\mu$ which have a corresponding $d-1$-degree Hamiltonian multivector.

In section B.3, 'Functional form of hamiltonian $d-1$-forms', the brackets $\{G,F\} := X_G \lrcorner X_F \lrcorner \Omega = X_G \lrcorner \mathrm{d}F$ are calculated directly using the most general functional form of the hamiltonian $d-1$-form. The structure of Ω strongly constrains the functional form of the components of the vector field X_F and the $d-1$ form F. This is investigated writing out the most general expansion of $X_F \lrcorner \Omega = \mathrm{d}F$ in a coordinate basis and then identifying the components on each side of the equation. This calculation of the multi-Poisson bracket may possibly serve as a definition of the brackets for the more general case of $d-1$-form observables which are not hamiltonian. This is the following:

Multi-Poisson brackets on covariant multiphase space

A natural generalization of the Poisson bracket to the covariant multiphase space setting is the directional bracket, an antisymmetric bilinear binary operation on functions on the covariant multiphase-space bundle over spacetime. This is the set, for $\alpha = 0 \ldots d-1$, of d Poisson brackets, which we call the multi-Poisson (or directional-Poisson) bracket:

$$\{f,g\}_\alpha := -\mathrm{d}_v f \llcorner \Pi_\alpha \lrcorner\, \mathrm{d}_v g = f \cdot \left(\frac{\overleftarrow{\partial}}{\partial u^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha}\right) \cdot g = \left(\frac{\partial f}{\partial u^i}\right)\left(\frac{\partial g}{\partial p_i^\alpha}\right) - \left(\frac{\partial f}{\partial p_i^\alpha}\right)\left(\frac{\partial g}{\partial u^i}\right) \tag{3.9}$$

where the exterior derivative d_v here is the vertical exterior derivative on the fiber of the covariant multiphase-space bundle over the spacetime base space. $\Pi_\alpha := (\frac{\overleftarrow{\partial}}{\partial u^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha})$ is an antisymmetric bivector field constructed on the multiphase space with coordinates (u^i, p_i^α).

This family of brackets parametrized by the spacetime directional index α can be written more geometrically as a horizontal 1-form $\{f,g\}_\alpha dx^\alpha$, or as a $d-1$ multivector acting as a biderivative in the vertical directions:

$$\{f,g\} \;:= {}^d\partial \llcorner \; \{f,g\}_\alpha dx^\alpha \;=\; (-1)^d \{f,g\}_\alpha dx^\alpha \lrcorner\, {}^d\partial = \{f,g\}_\alpha \partial^\alpha \tag{3.10}$$

where

$$\partial^\alpha \;:=\; \frac{\partial}{\partial x_\alpha} \;:=\; (-1)^\alpha \frac{\partial}{\partial x^{d-1}} \wedge \frac{\partial}{\partial x^{d-2}} \wedge \ldots \wedge \widehat{\frac{\partial}{\partial x^\alpha}} \wedge \ldots \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^0} \;=$$

$$(-1)^{d-(\alpha+1)}\frac{\partial}{\partial x^0}\wedge\frac{\partial}{\partial x^1}\wedge\frac{\partial}{\partial x^2}\wedge\ldots\wedge\widehat{\frac{\partial}{\partial x^\alpha}}\wedge\ldots\wedge\frac{\partial}{\partial x^{d-2}}\wedge\frac{\partial}{\partial x^{d-1}}={}^d\partial\llcorner\mathrm{d}x^\alpha=(-1)^d\,\mathrm{d}x^\alpha\lrcorner{}^d\partial \tag{3.11}$$

$\widehat{\frac{\partial}{\partial x^\alpha}}$ means that the factor $\frac{\partial}{\partial x^\alpha}$ is omitted from the exterior product.

[See section 1.3 for conventions on symbols in differential geometry in use in this thesis.]

The multi bracket may be written as a multivector operator which acts as a derivative in the fiber directions on a pair of functions f, g:

$$\begin{aligned}\{f,g\} &:= f\,\overleftarrow{\mathrm{d}_v}\llcorner\Pi\lrcorner\overrightarrow{\mathrm{d}_v}g=-\mathrm{d}_vf\llcorner\Pi\lrcorner\mathrm{d}_vg=f\cdot\;\Pi\;\cdot g\\ &:= f\cdot(\frac{\overleftarrow{\partial}}{\partial u^i}\wedge\frac{\partial}{\partial x_\alpha}\wedge\frac{\overrightarrow{\partial}}{\partial p_i^\alpha})\cdot g\end{aligned} \tag{3.12}$$

If f or g are spacetime forms then they contract with the spacetime multivector factors of the multi-Poisson bracket. The $d+1$-multivector field which acts on the pair of function is $\Pi := (\frac{\overleftarrow{\partial}}{\partial u^i}\wedge\frac{\partial}{\partial x_\alpha}\wedge\frac{\overrightarrow{\partial}}{\partial p_i^\alpha})$.

Some more detail on this bracket, including the definition of brackets with grassmann odd coordinates on multiphase space is in appendix B.1.

DeDonder-Weyl equations expressed using brackets

With the multi-Poisson brackets, the DeDonder-Weyl (DDW) equations of motion (3.48) (which are the analogue of the Hamilton's equations of motion and are the multiphase space equations of motion for fields) may be written in a similar way as Hamilton's equations with Poisson brackets. We do this showing the various ways this may be written in the notation of differential geometry.

The first DDW equation is

$$\partial_\alpha u^i\approx\{u^i,\mathfrak{H}\}_\alpha=\frac{\partial\mathfrak{H}}{\partial p_i^\alpha} \tag{3.13}$$

which can be written as

$$\mathrm{d}u^i\;=\;\partial_\alpha u^i\mathrm{d}x^\alpha\approx\{u^i\,,\,\mathfrak{H}\,\mathrm{d}^dx\}=\{u^i\,,\,\underline{\mathfrak{H}}\,\}=\frac{\partial\mathfrak{H}}{\partial p_i^\alpha}\mathrm{d}x^\alpha\;=\;\partial^\alpha\wedge\frac{\partial}{\partial p_i^\alpha}\lrcorner\,\mathrm{d}\underline{\mathfrak{H}} \tag{3.14}$$

The second DDW equation is

$$\delta_\beta^\beta\;\partial_\alpha p_i^\alpha\approx\{p_i^\alpha,\mathfrak{H}\}_\alpha=\;-\;\frac{\partial\mathfrak{H}}{\partial u^i}\;\delta_\alpha^\alpha \tag{3.15}$$

which can be written as

$$\delta_\beta^\beta\;\mathrm{d}\underline{p}_i=\;\;\delta_\beta^\beta\;\mathrm{d}(p_i^\alpha\mathrm{d}x_\alpha)\;\;=\;\;\delta_\beta^\beta\;\partial_\alpha p_i^\alpha\mathrm{d}^dx\;\;\approx\;\;\{p_i^\alpha,\mathfrak{H}\}_\alpha\mathrm{d}^dx\;\;=\;\;d\{p_i^\alpha\mathrm{d}x_\alpha\,,\,\mathfrak{H}\,\mathrm{d}^dx\}$$

$$= d\{\underline{p}_i\,,\,\underline{\mathfrak{H}}\} \quad = d\{p_i^\alpha \mathrm{d}x_\alpha\,,\,\mathfrak{H}\}\,\mathrm{d}^d x \;=\; -\frac{\partial\mathfrak{H}}{\partial u^i}\mathrm{d}^d x\;\delta^\alpha_\alpha \quad = \; -\frac{\partial\underline{\mathfrak{H}}}{\partial u^i}\;\delta^\alpha_\alpha \quad = \; -\frac{\partial}{\partial u^i}\,\lrcorner\,\mathrm{d}\underline{\mathfrak{H}}\;\delta^\alpha_\alpha \tag{3.16}$$

In $\mathrm{d}u^i$ and $\mathrm{d}\underline{p}_i$, the exterior derivative d is acting on a section in the multiphase space, i.e. functions or forms on spacetime, representing a particular covariant multiphase field configuration which satisfies the DDW equations. The above are the DeDonder Weyl equations of motion for such field configurations.

Summarizing, to emphasize the parallel with Hamilton's equations of motion expressed using Poisson brackets, we succinctly write the DDW equations as 'multi-Hamiltonian equations of motion':

$$\mathrm{d}u^i \approx \;\{u^i\,,\,\underline{\mathfrak{H}}\}\quad\text{and}\quad \mathrm{d}\underline{p}_i \;\approx\; \{\underline{p}_i\,,\,\underline{\mathfrak{H}}\} \tag{3.17}$$

However, unlike in Hamilton's mechanics, one cannot write $\mathrm{d}\mathcal{O} \approx \;\{\mathcal{O}\,,\,\underline{\mathfrak{H}}\}$ for arbitrary observables $\mathcal{O}$.

Combining left hand sides of these equations into one expression:

$$\begin{aligned} &-\;[\partial_\alpha p_i^\alpha(x)]\,\mathrm{d}u^i\wedge\mathrm{d}^d x + [\partial_\alpha u^i(x)]\,\mathrm{d}p_i^\alpha\wedge\mathrm{d}^d x \\ =\;&(\;[\partial_\alpha p_i^\gamma(x)]\,\mathrm{d}x^\alpha\wedge\frac{\partial}{\partial p_i^\gamma}\;+\;[\partial_\alpha u^i(x)]\,\mathrm{d}x^\alpha\wedge\frac{\partial}{\partial u^i})\;\stackrel{\wedge}{\lrcorner}\;\mathrm{d}p_j^\beta\wedge\mathrm{d}u^j\wedge\mathrm{d}x_\beta \\ =\;&(\;[\partial_\alpha Z^K(x)]\,\mathrm{d}x^\alpha\wedge\frac{\partial}{\partial Z^K}\;)\;\stackrel{\wedge}{\lrcorner}\;\Omega \end{aligned} \tag{3.18}$$

where Z^K are the coordinates of the fibers over spacetime of covariant multiphase space. Ω is the multisymplectic form.

Now combining right hand sides of these equations into one expression:

$$\frac{\partial\mathfrak{H}}{\partial u^i}\,\mathrm{d}u^i\wedge\mathrm{d}^d x\;+\;\frac{\partial\mathfrak{H}}{\partial p_i^\alpha}\,\mathrm{d}p_i^\alpha\wedge\mathrm{d}^d x\quad\approx\mathrm{d}\,\underline{\mathfrak{H}} \tag{3.19}$$

We have the DeDonderWeyl equations of motion:

$$(\;[\partial_\alpha Z^K(x)]\,\mathrm{d}x^\alpha\wedge\frac{\partial}{\partial Z^K}\;)\;\stackrel{\wedge}{\lrcorner}\;\Omega\;\approx\;\mathrm{d}\,\underline{\mathfrak{H}} \tag{3.20}$$

Algebraic properties

The multi-Poisson bracket is antisymmetric:-

$$\{A,B\}_\alpha\mathrm{d}x^\alpha = -\{B,A\}_\alpha\mathrm{d}x^\alpha \tag{3.21}$$

Bilinear for coefficients constant on the multiphase-space fiber:-

$$\{k_1A,k_2B\}_\alpha\mathrm{d}x^\alpha = k_1k_2\{A,B\}_\alpha\mathrm{d}x^\alpha \tag{3.22}$$

Has the Leibnitz rule for products of functions on multiphase space:-

$$\{A, BC\}_\alpha \mathrm{d}x^\alpha = C\{A, B\}_\alpha \mathrm{d}x^\alpha + B\{A, C\}_\alpha \mathrm{d}x^\alpha \tag{3.23}$$

The Jacobi identity:-

Given two binary operations $\{\cdot\ ,\ \cdot\}_\alpha$, $\{\cdot\ ,\ \cdot\}_\beta$, we define the jacobiator $J_{\alpha\beta}(\cdot\ ,\ \cdot\ ,\ \cdot\ )$ by:

$$J_{\alpha\beta}(A, B, C) := \ \{\{A, B\}_\alpha, C\}_\beta + \ \{\{B, C\}_\alpha, A\}_\beta + \ \{\{C, A\}_\alpha, B\}_\beta \tag{3.24}$$

The jacobiator with the multi-Poisson brackets of a cyclic triple (A, B, C) of functions on multiphase space is

$$J_{\alpha\beta}(A, B, C) := \ \{\{A, B\}_\alpha, C\}_\beta + \ \{\{B, C\}_\alpha, A\}_\beta + \ \{\{C, A\}_\alpha, B\}_\beta \tag{3.25}$$

If $\alpha = \beta$ are the same direction then we recover the Jacobi identity on Poisson brackets $J_{\alpha\alpha}(A, B, C) = 0$. For $g^{\alpha\beta}$ a symmetric tensor, $g^{\alpha\beta} J_{\alpha\beta}(A, B, C) = \ 0$. So $J_{\alpha\beta}(A, B, C)$ are components of a spacetime horizontal 2-form, $J(A, B, C) := \mathrm{d}x^\alpha \wedge \mathrm{d}x^\beta J_{\alpha\beta}(A, B, C)$.

3.2.1 Extended multi-Poisson brackets on multiphase space

Now we look at the multi-Poisson bracket defined on *multiphase space* which has an extra coordinate p, as opposed to covariant multiphase space above.

A natural generalization of the Poisson bracket in the multiphase space is the directional bracket, an antisymmetric bilinear binary operation on functions on the multiphase-space bundle over spacetime. This is the sequence, for $\alpha = 0 \ldots d-1$, of d brackets, which we call the extended multi Poisson (or directional Poisson) bracket on multiphase space:

$$\widetilde{\{f, g\}}_\alpha := -\mathrm{d}f \llcorner \widetilde{\Pi}_\alpha \lrcorner \, \mathrm{d}g = f \cdot (\frac{\overleftarrow{\partial}}{\partial u^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha} + \frac{\overleftarrow{\partial}}{\partial x^\alpha} \wedge \frac{\overrightarrow{\partial}}{\partial p}) \cdot g \ =$$

$$= \left(\frac{\partial f}{\partial u^i}\right)\left(\frac{\partial g}{\partial p_i^\alpha}\right) - \left(\frac{\partial f}{\partial p_i^\alpha}\right)\left(\frac{\partial g}{\partial u^i}\right) + \left(\frac{\partial f}{\partial x^\alpha}\right)\left(\frac{\partial g}{\partial p}\right) - \left(\frac{\partial f}{\partial p}\right)\left(\frac{\partial g}{\partial x^\alpha}\right) \tag{3.26}$$

This family of brackets parametrized by the spacetime directional index α can be written more geometrically as a horizontal 1-form $\widetilde{\{f, g\}}_\alpha dx^\alpha$ or a $d-1$ multivector:

$$\widetilde{\{f, g\}} \ := {}^d\partial \llcorner \ \widetilde{\{f, g\}}_\alpha \mathrm{d}x^\alpha \ = \ (-1)^d \widetilde{\{f, g\}}_\alpha \mathrm{d}x^\alpha \lrcorner \, {}^d\partial = \widetilde{\{f, g\}}_\alpha \partial^\alpha \tag{3.27}$$

In this bracket the spacetime derivatives need to be viewed as underspecified:

Because multiphase-space is a bundle over spacetime, the partial derivative with respect to spacetime is not fully defined, because the notion of a horizontal direction $\frac{\partial}{\partial x^\alpha}$, with u^i, p_i^α, and p constant, is not defined - unless we have a connection. So we cannot write $\widetilde{\{}u^i,\cdot\} = \frac{\partial}{\partial p_i^\alpha}$, because we cannot assume $\frac{\partial u^i}{\partial x^\alpha} = 0$. Rather we write $\widetilde{\{}u^i,\cdot\} = \frac{\partial}{\partial p_i^\alpha} + (\partial^1_\alpha u^i)\frac{\partial}{\partial p}$, where the horizontal direction ∂^1_α which projects to ∂_α vector on the base space is yet to be determined. This is seen below where u^i is simply the fixed coordinate function on multiphase space and solving the bracket equation for the solution $u^i(x)$ from $\partial_\alpha u^i$ amounts to finding the horizontal direction ∂^1_α.

With the multi Poisson brackets, the DeDonder-Weyl equations of motion may be written in a similar way as Hamilton's equations are using Poisson brackets, with the Hamiltonian constraint function $H = \mathcal{H} - p$ (employing various notations) :

The first DeDonder-Weyl equation (DDW1):

$$0 \approx \widetilde{\{}u^i, H\}_\alpha = \widetilde{\{}u^i, \mathfrak{H} - p\}_\alpha = \frac{\partial u^i}{\partial u^j}\frac{\partial \mathfrak{H}}{\partial p_j^\alpha} - \frac{\partial u^i}{\partial^1 x^\alpha}\frac{\partial p}{\partial p} = \frac{\partial \mathfrak{H}}{\partial p_i^\alpha} - \partial^1_\alpha u^i \quad \leftrightarrow \tag{3.28}$$

$$0 \approx \widetilde{\{}u^i, \underline{H}\} = \widetilde{\{}u^i, (\mathfrak{H} - p)\mathrm{d}^d x\} = \{u^i\,,\, \mathfrak{H}\,\mathrm{d}^d x\} - \partial^1_\alpha u^i \mathrm{d}x^\alpha = \{u^i\,,\, \underline{\mathfrak{H}}\,\} - \mathrm{d}^1 u^i =$$

$$= \frac{\partial \mathfrak{H}}{\partial p_i^\alpha}\mathrm{d}x^\alpha - \partial^1_\alpha u^i \mathrm{d}x^\alpha = \partial^\alpha \wedge \frac{\partial}{\partial p_i^\alpha} \lrcorner\, \mathrm{d}\underline{\mathfrak{H}} - \mathrm{d}^1 u^i \tag{3.29}$$

The second DeDonder-Weyl equation (DDW2):

$$0 \approx \widetilde{\{}p_i^\alpha, H\}_\alpha = \{p_i^\alpha, \mathfrak{H} - p\}_\alpha = -\frac{\partial p_i^\alpha}{\partial p_j^\alpha}\frac{\partial \mathfrak{H}}{\partial u^j} - \frac{\partial p_i^\alpha}{\partial^2 x^\alpha}\frac{\partial p}{\partial p} = -\frac{\partial \mathfrak{H}}{\partial u^i}\,\delta^\alpha_\alpha - \delta^\beta_\beta\,\partial^2_\alpha p_i^\alpha \quad \leftrightarrow \tag{3.30}$$

$$0 \approx \widetilde{\{}p_i^\alpha, H\}_\alpha \mathrm{d}^d x = \{p_i^\alpha, \mathfrak{H}\}_\alpha \mathrm{d}^d x - \delta^\beta_\beta\,\partial^2_\alpha p_i^\alpha \mathrm{d}^d x = \widetilde{\{}p_i^\alpha \mathrm{d}x_\alpha\,,\, H\,\mathrm{d}^d x\} =$$

$$= \{p_i^\alpha \mathrm{d}x_\alpha\,,\, \mathfrak{H}\,\mathrm{d}^d x\} - \delta^\beta_\beta\,\mathrm{d}^2(p_i^\alpha \mathrm{d}x_\alpha) = \widetilde{\{}\underline{p}_i\,,\, \underline{H}\,\} = \{\underline{p}_i\,,\, \underline{\mathfrak{H}}\,\} - \delta^\beta_\beta\,\mathrm{d}^2 \underline{p}_i =$$

$$= -\frac{\partial \mathfrak{H}}{\partial u^i}\mathrm{d}^d x\,\delta^\alpha_\alpha - \delta^\beta_\beta\,\partial^2_\alpha p_i^\alpha \mathrm{d}^d x = -\frac{\partial \underline{\mathfrak{H}}}{\partial u^i}\,\delta^\alpha_\alpha - \delta^\beta_\beta\,\mathrm{d}^2(p_i^\alpha \mathrm{d}x_\alpha) =$$

$$= -\frac{\partial}{\partial u^i} \lrcorner\, \mathrm{d}\underline{\mathfrak{H}}\,\delta^\alpha_\alpha - \delta^\beta_\beta\,\mathrm{d}^2 \underline{p}_i \tag{3.31}$$

In these brackets, H is viewed as a specified function on multiphase space, and so are u^i and p_i^μ just coordinate functions on multiphase space, rather than functions on spacetime representing solutions to the DDW equations. The equations above can be thought as determining the connection ∂^1_α which specifies the actual spacetime derivatives $\partial_\alpha u^i$ of solutions to the DDW1 equations and the the connection ∂^2_α which specifies the actual spacetime derivatives $\partial_\alpha p_i^\alpha$ of solutions to the DDW2 equations. If, in the brackets above, we had used the partial derivatives ∂_α, keeping u^i and p_i^α constant, then of course we would have $\partial_\alpha u^i$ and $\partial_\alpha p_i^\alpha = 0$ when calculating the brackets with the coordinate functions u^i and p_i^μ.

Summarizing, to emphasize the parallel with Hamiltons equations of motion, we succinctly write the 'multi-Hamiltonian equations of motion':

$$0 \approx \widetilde{\{} u^i \, , \underline{H} \} \quad \text{and} \quad 0 \approx \widetilde{\{} \underline{p}_i \, , \underline{H} \} \tag{3.32}$$

3.2.2 Multiphase-space brackets of $d-1$-forms

On a multisymplectic manifold we can define the skew symmetric *Kanantchikov bracket* [47] between two hamiltonian $d-1$-forms f and g, with corresponding hamiltonian vectorfields X_f, X_g, similarly to (2.13):

$$\{f, g\} = -X_f \lrcorner X_g \lrcorner \Omega = -X_f \lrcorner \mathrm{d}g = X_g \lrcorner \mathrm{d}f = (-1)^d \mathrm{d}f \llcorner X_g = (-1)^d X_f \lrcorner \Omega \llcorner X_g \tag{3.33}$$

Note that $\{f, g\}$ is a $d-1$ form. This bracket has the Lie algebra homomorphism property

$$[X_f, X_g] \lrcorner \Omega = X_{\{f,g\}} \lrcorner \Omega \tag{3.34}$$

and the Jacobiator is

$$J_{\{\}}(f, g, h) := \{\{f, g\}, h\} + \{\{g, h\}, f\} + \{\{h, f\}, g\} = \mathrm{d}(X_f \lrcorner X_g \lrcorner X_h \lrcorner \Omega) \tag{3.35}$$

The Jacobiator is an exact $d-1$ form, so $(\tilde{\mathcal{H}}^{d-1}(\mathcal{M}, \Omega), \{,\})$ is a Lie algebra isomorphic to $(\mathrm{Ham}^1(\mathcal{M}, \Omega), [,])$. Note that the associative product of spacetime horizontal $d-1$ forms over a d dimensional spacetime is zero: $f \wedge g = 0$ so, as a Poisson algebra, the associative product is trivial.

3.3 Multiphase space Lagrangian field theory

3.3.1 Legendre transformation

A Lagrangian density $\mathbb{L}$ which is first order in spacetime derivatives of the dynamical fields, such as is used to specify the action of a field (as examined in section 2.3) by

$$S[u^i(x)] = \int_B \mathcal{L}(x^\mu, u^i, u^i_{,\mu}) \; \mathrm{d}^d x = \int_B \mathbb{L}(x^\mu, u^i, u^i_{,\mu}) \tag{3.36}$$

generates a Legendre transformation [28], called the extended covariant Legendre transformation,

$$\hat{\mathbb{F}\mathcal{L}} : J^1\mathcal{E} \longrightarrow P_{\mathbb{L}} \subset \mathcal{M} :: (x^\mu, u^i, u^i_\mu) \longmapsto (x^\mu, u^i, p^\mu_i, p) = (x^\mu, u^i, \frac{\partial \mathcal{L}}{\partial u^i_\mu}, \mathcal{L} - \left(\frac{\partial \mathcal{L}}{\partial u^i_\mu}\right) u^i_\mu) \tag{3.37}$$

where $P_{\mathbb{L}}$ is the image, which is a bundle map over $\mathcal{E}$ from the jet bundle to the multiphase space. For a Lagrangian density $\mathcal{L}(x^\mu, u^i, u^i_\mu)$, the Legendre transformation maps points

$v = u^i_\mu$ in the fiber of first jet space to the points $\hat{\mathbb{F}\mathcal{L}}(v) = \pi = (p^\mu_i, p)$ in the fiber of the dual jet space such that $\hat{\mathbb{F}\mathcal{L}}(v)$ is the affine approximation to $\mathbb{L}$ at v:

$$\langle\ \hat{\mathbb{F}\mathcal{L}}(v), W\ \rangle \ = \ \mathbb{L}(x^\mu, u^i, v) + \frac{\partial}{\partial s}\mathbb{L}(x^\mu, u^i, v + s(W - v))|_{s=0}$$

This can be written with local coordinates as

$$\langle\ \pi, W\ \rangle \ = \ p + p^\mu_i W^i_\mu \ = \ \mathcal{L}(x^\mu, u^i, u^i_\mu) + \frac{\partial \mathcal{L}}{\partial u^i_\mu}(W^i_\mu - u^i_\mu)\mathrm{d}^d x \tag{3.38}$$

In the regular case, $\mathcal{P}_{\mathbb{L}}$ is a codimension 1 submanifold, $\bar{H}$, of $\mathcal{M}$ defined by the equation $p = -\mathcal{H}$, where $\mathcal{H}(x^\mu, u^i, p^\mu_i) = \mathcal{L} - \left(\frac{\partial \mathcal{L}}{\partial u^i_\mu}\right) u^i_\mu$ as a function on $\mathcal{P}_{\mathbb{L}}$. The pull back of the canonical forms on the multiphase space $\mathcal{M}$, $\widetilde{\Theta} = \mathrm{d}u^i \wedge\ p^\alpha_i \mathrm{d}^{d-1}x_\alpha + p\ \mathrm{d}^d x$ and $\widetilde{\Omega} = -\mathrm{d}\widetilde{\Theta}$, to the first jet bundle $J^1\mathcal{E}$ via $\hat{\mathbb{F}\mathcal{L}}$ are respectively:

$$\widetilde{\Theta}_{\mathcal{L}} = \hat{\mathbb{F}\mathcal{L}}^*\widetilde{\Theta} = \left(\frac{\partial \mathcal{L}}{\partial u^i_\mu}\right)\mathrm{d}u^i \wedge \mathrm{d}^{d-1}x_\mu + (\mathcal{L} - \left(\frac{\partial \mathcal{L}}{\partial u^i_\mu}\right) u^i_\mu)\ \mathrm{d}^d x \quad \text{and} \quad \widetilde{\Omega}_{\mathcal{L}} = \hat{\mathbb{F}\mathcal{L}}^*\widetilde{\Omega} = -\mathrm{d}\widetilde{\Theta}_{\mathcal{L}} \tag{3.39}$$

This endows the jet bundle with a multisymplectic (in the regular case) or pre-multisymplectic (in the non-regular case) d and $d+1$ forms just as in symplectic mechanics the Legendre transformation pulls back the canonical symplectic form from the momentum phase space to the velocity phase space.

Similarly to symplectic mechanics, the integral of the multisymplectic d-form, $\widetilde{\Theta}_{\mathcal{L}}$ above, on the jet bundle over a spacetime field configuration (a prolonged section $j^1\phi : B \longrightarrow \mathcal{E} \longrightarrow J^1\mathcal{E}$ on $\pi^{B,J^1\mathcal{E}}$), gives the action of the field:

$$\int_{j^1\phi} \widetilde{\Theta}_{\mathcal{L}} = \int_B j^1\phi^* \widetilde{\Theta}_{\mathcal{L}} = \int_B \mathcal{L}(j^1\phi)\ \mathrm{d}^d x = S[\phi] \tag{3.40}$$

and the solution of the variational problem

$$\delta_\xi \int_{j^1\phi} \widetilde{\Theta}_{\mathcal{L}} = \delta_\xi \int_B \mathcal{L}(j^1\phi)\mathrm{d}^d x = \delta_\xi S[\phi] = 0 \tag{3.41}$$

is given by

$$\delta_\xi \mathcal{L}(j^1\phi) = j^1\phi^*(\delta_\xi \widetilde{\Theta}_{\mathcal{L}}) = j^1\phi^*(\xi \lrcorner \mathrm{d}\widetilde{\Theta}_{\mathcal{L}}) = 0 \tag{3.42}$$

If we set $\xi = \left(\frac{\partial}{\partial u^i}\right)$ in $j^1\phi^*(\xi \lrcorner \mathrm{d}\widetilde{\Theta}_{\mathcal{L}}) = 0$ we obtain, in local coordinates, the Euler-Lagrange equations of motion of the field $\phi(x^\mu)$:

$$\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial u^i_\mu}\right) - \left(\frac{\partial \mathcal{L}}{\partial u^i}\right) = 0, \quad u^i(x^\mu) = \phi^i(x^\mu), \quad u^i_\nu(x^\mu) = \frac{\partial \phi^i}{\partial x^\nu}(x^\mu) \tag{3.43}$$

In fact, the action $S[u^i(x)]$ of a field prolonged section (=path) $C_{J^1\mathcal{E}} = (u^i(x), \partial_\mu u^i(x))$ in $J^1\mathcal{E}$ is the integral of the extended canonical d-form $\widetilde{\Theta}$ over the constrained section $C_{\bar{H}} = \mathbb{F}\tilde{\mathcal{L}}(C_{J^1\mathcal{E}})$ inside the hypersurface $\bar{H}$ embedded in the multiphase space, and of the d-form $(p^\mu_i \partial_\mu u^i - \mathcal{H})\ \mathrm{d}^d x$ over the path $C_{\mathcal{M}} = F\tilde{\mathcal{L}}(C_{J^1\mathcal{E}})$ in multiphase space, and of the $d+1$ -form

$\mathrm{d}\mathbb{L} = [\left(\frac{\partial L}{\partial u^i}\right) \mathrm{d}u^i + \left(\frac{\partial \mathcal{L}}{\partial u^i_\mu}\right) \mathrm{d}u^i_\mu] \wedge \mathrm{d}^d x$ over the section $C_{J^1\mathcal{E}}$ in the jet bundle, and (by defintion) of the d-form $\mathcal{L}\ \mathrm{d}^d x$ over the section $C_\mathcal{E}$ in configuration space :

$$\int_{C_{\bar{H}}} \widetilde{\Theta} = \int (\ p^\mu_i \partial_\mu u^i + p\ ) \mathrm{d}^d x \llcorner \bar{X}\big|_{C_{\bar{H}}} = \int_{C_\mathcal{M}} (p^\mu_i \partial_\mu u^i - \mathcal{H}(x, u^i, p^\mu_i))\, \mathrm{d}^d x = \int_{C_{J^1\mathcal{E}}} \mathcal{L}\, \mathrm{d}^d x = S[C_\mathcal{E}] \tag{3.44}$$

An infinitesimal variation $Y_{\bar{H}}$ of the section $C_{\bar{H}} = \mathbb{F}\tilde{\mathcal{L}}(C_{J^1\mathcal{E}})$ inside $\bar{H}$ in extended phase space changes the action by

$$\begin{aligned} \delta_Y S[C_\mathcal{E}] &= \delta_Y \int_{C_{J^1\mathcal{E}}} \mathcal{L}\, \mathrm{d}^d x = \delta_{Y_{\bar{H}}} \int_{C(\bar{H})} \widetilde{\Theta} = \int_{C(\bar{H})} \mathcal{L}_{Y_{\bar{H}}} \widetilde{\Theta} = \int_{C(\bar{H})} (\mathrm{d}i_{Y_{\bar{H}}} + i_{Y_{\bar{H}}} \mathrm{d}) \widetilde{\Theta} \\ &= \int_{C(\bar{H})} \mathrm{d}i_{Y_{\bar{H}}} \widetilde{\Theta} + i_{Y_{\bar{H}}} \widetilde{\Omega} = \int_{\partial C(\bar{H})} i_{Y_{\bar{H}}} \widetilde{\Theta} + \int_{C(\bar{H})} i_{Y_{\bar{H}}} \widetilde{\Omega} \end{aligned} \tag{3.45}$$

If the variation vector field is zero, $Y_{\bar{H}} = 0$, on the boundary $\partial C(\bar{H})$ of the section $C(\bar{H})$, we obtain

$$\delta_Y S[C_\mathcal{E}] = \delta_Y \int_{C_{J^1\mathcal{E}}} \mathcal{L}\, \mathrm{d}^d x = \delta_{Y_{\bar{H}}} \int_{C(\bar{H})} \widetilde{\Theta} = \int_{C(\bar{H})} i_{Y_{\bar{H}}} \widetilde{\Omega} = - \int_{C(\bar{H})} X_{\bar{H}} \lrcorner Y_{\bar{H}} \lrcorner \widetilde{\Omega}\, \mathrm{d}^d x \tag{3.46}$$

where $X_{\bar{H}}$ is the tangent multivector of the time-parametrized path in multiphase space (the reason for writing $X_{\bar{H}}$ explicitly in the integral is made clear in the next sentence).

If, in addition, the symplectic equations of motion, $i_{X_{\bar{H}}} \widetilde{\Omega} \approx 0$, in extended phase space hold for the trajectory $C(\bar{H})$, then the integrand is zero for any infinitesimal variation $Y_{\bar{H}}$ of the path, whose tangent multivector is $X_{\bar{H}}$:

$$\delta_Y S[C_\mathcal{E}] = \delta_Y \int_{C_{J^1\mathcal{E}}} \mathcal{L}\, \mathrm{d}^d x = \delta_{Y_{\bar{H}}} \int_{C(\bar{H})} \widetilde{\Theta} = \int_{C(\bar{H})} Y_{\bar{H}} \lrcorner (X_{\bar{H}} \lrcorner \widetilde{\Omega})\, \mathrm{d}^d x \approx 0 \tag{3.47}$$

and so the action is at a stationary point when $X_{\bar{H}} \lrcorner \widetilde{\Omega} \approx 0$, as shown in the appendix E (Multivector picture).

3.4 Multisymplectic Hamiltonian field theory

The analogue of the Hamiltonian function on phase space and Hamilton's equations of motion is the DeDonder-Weyl 'Hamiltonian' on covariant multiphase space and the DeDonder-Weyl field equations. The DDW Hamiltonian is obtained from a Lagrangian density as shown in section 3.3.1: The DDW Hamiltonian is $\mathcal{H} = \left(\frac{\partial \mathcal{L}}{\partial u^i_\mu}\right) u^i_\mu - \mathcal{L}$ as a function on covariant multiphase space.

3.4.1 DeDonder Weyl (DDW) equations

The Legendre transformation (3.3.1) produces a DeDonder-Weyl (DDW) 'Hamiltonian' density $\mathcal{H}(x,u^i,p_i^\mu)$ on $P_{\mathcal{L}}$. For a regular transformation, solutions $u^i(x)$ to the second order Euler-Lagrange equations for the configuration space action $S[u^i(x)]$ are solutions to the following first order (in spacetime derivatives) DeDonder-Weyl equations:

$$\begin{aligned}\partial_\mu u^i(x) - \frac{\partial\mathcal{H}}{\partial p_i^\mu}(x,u^i(x),p_i^\mu(x)) = 0 \qquad (DDW1)\\ \partial_\mu p_i^\mu(x) + \frac{\partial\mathcal{H}}{\partial u^i}(x,u^i(x),p_i^\mu(x)) = 0 \qquad (DDW2)\end{aligned} \tag{3.48}$$

where $p_i^\mu(x)$ is the Legendre transformation of the prolongation of $u^i(x)$.

The DeDonder Weyl equations are the Euler Lagrange equations for a first order Lagrangian density for a field in the same way as Hamilton's equations of motion are for a first order Lagrangian for a mechanical system (2.31) . The derivation of the DeDonder Weyl equations from first order action is given in section 3.4.2.

Solutions of DDW equations viewed as a connection on covariant multiphase space.

Expressed more rigorously in the language of bundles, the DDW Hamiltonian 'function' on the dual jet bundle is a section $\Gamma_H : J^1\mathcal{E}^* \longrightarrow \mathcal{M}$. In local coordinates, a section

$$(x^\mu,u^i,p_i^\mu,p) \;=\; \Gamma_H(x^\mu,u^i,p_i^\mu) \;=\; (x^\mu,u^i,p_i^\mu,-\mathcal{H}(x^\mu,u^i,p_i^\mu)) \tag{3.49}$$

of the bundle $\pi^{J^1E^*,\mathcal{M}} : \mathcal{M} \simeq \Lambda_1^d \longrightarrow \frac{\Lambda_1^d}{\Lambda_0^d} \simeq J^1\mathcal{E}^*$. The base space is the covariant multiphase space above. Given the Lagrangian d-form $\mathbb{L} = L\mathrm{d}^dx$, the Legendre transformation gives the DDW Hamiltonian d-form $\mathcal{H}(x^\mu,u^i,p_i^\mu)\ \mathrm{d}^dx = [\left(\frac{\partial\mathcal{L}}{\partial u_\mu^i}\right)(u_\mu^i + \mathfrak{U}_\mu^i) - L]\ \mathrm{d}^dx$. The $\mathfrak{U}_\mu^i$ is a connection on the bundle $\pi^{B,\mathcal{E}}$, required because the DDW Hamiltonian function may only be locally defined if the bundle $\pi^{J^1\mathcal{E}^*,\mathcal{M}}$ is non-trivial.

Ω_H is the multisymplectic form on $\mathcal{M}$ pulled back to $J^1\mathcal{E}^*$ via Γ_H, $\Omega_H = \Gamma_H^*\Omega = -\mathrm{d}(-\mathcal{H}\mathrm{d}^dx + p_i^\mu\mathrm{d}u^i \wedge \mathrm{d}^{d-1}x_\mu)$. An Ehresmann connection E on some subset $C \subset J^1\mathcal{E}^*$ of the bundle $\pi^{J^1\mathcal{E}^*,B} : J^1\mathcal{E}^* \longrightarrow B$, may be viewed as a degree d horizontal multivector field $\bar{X}_E$ defined on the subset C. If the equation $\bar{X}_E \lrcorner\, \Omega_H = 0$ holds in some subset $C \subset J^1\mathcal{E}^*$, then $\bar{X}_E$ is said to be the generalized Hamiltonian connection for $\mathcal{H}$ on the set C.

If a section v of the bundle $\pi^{J^1\mathcal{E}^*,B} : J^1\mathcal{E}^* \longrightarrow B$, $v(x) = (x^\mu, v^i(x), v_i^\mu(x))$, has a tangent space which is horizontal in E and satisfies $\bar{X}_E \lrcorner\, \Omega_H = 0$ on $C \subset J^1\mathcal{E}^*$, then the section satisfies the DDW equations on C: $\frac{\partial v(x)}{\partial x^\mu} = \frac{\partial}{\partial x^\mu}(x^\mu, v^i(x), v_i^\mu(x)) \approx (d, \frac{\partial\mathcal{H}}{\partial p_i^\mu}, -\frac{\partial\mathcal{H}}{\partial v^i})$. Clearly the Ehresmann connection E must be flat (integrable) on that part of the section which is in C (see appendix E).

3.4.2 Multiphase-space action

The covariant *multiphase-space action* defined to be:

$$S_{MP}[u^i(x), p_i^\mu(x)] \ := \int_{\Gamma J^1\mathcal{E}^*} \mathcal{L}_{MP}\ \mathrm{d}^d x := \int_{\Gamma J^1\mathcal{E}^*} (p_i^\mu \partial_\mu u^i - \mathcal{H})\ \mathrm{d}^d x \tag{3.50}$$

where $\mathcal{H} = \mathcal{H}(x, u^i, p_i^\mu)$ is the DDW Hamiltonian.

The variation of the covariant multiphase-space action due to a general infinitesimal variation, $\delta p_i^\mu(x), \delta u^i(x), \delta x^\mu(x)$, of the fields, multimomenta, and coordinates of a partial section $\Gamma J^1\mathcal{E}^*$ of the affine dual of the jet bundle is:

$$\begin{aligned}
\delta S_{MP} \ &= \ \delta \int_{\Gamma J^1\mathcal{E}^*} (p_i^\mu \partial_\mu u^i - \mathcal{H}) \mathrm{d}^d x \\
= \int_{\Gamma J^1\mathcal{E}^*} [\, (\partial_\mu u^i - \frac{\partial \mathcal{H}}{\partial p_i^\mu})\delta p_i^\mu - (\partial_\mu p_i^\mu + \frac{\partial \mathcal{H}}{\partial u^i})\delta u^i + \frac{\partial \mathcal{H}}{\partial x^\mu}\delta x^\mu \ &+ \ \partial_\mu (\, \delta u^i p_i^\mu + (p_i^\kappa \partial_\kappa u^i - \mathcal{H})\delta x^\mu \,) \,]\ \mathrm{d}^d x \\
&= \int_{\Gamma J^1\mathcal{E}^*} [\, (\partial_\mu u^i - \frac{\partial \mathcal{H}}{\partial p_i^\mu})\delta p_i^\mu - (\partial_\mu p_i^\mu + \frac{\partial \mathcal{H}}{\partial u^i})\delta u^i \ + \ \frac{\partial \mathcal{H}}{\partial x^\mu}\delta x^\mu \,]\ \mathrm{d}^d x \\
&\quad + \ \int_{\partial \Gamma J^1\mathcal{E}^*} (\, \delta u^i p_i^\mu \ + (p_i^\kappa \partial_\kappa u^i - H)\delta x^\mu \,) \mathrm{d}S_\mu \\
&=: \int_{\Gamma J^1\mathcal{E}^*} [\, E_{p_i^\mu}\delta p_i^\mu \ + \ E_{u^i}\delta u^i \ + \ \mathcal{H}_\mu \delta x^\mu \ + \ \partial_\mu (\, \delta u^i p_i^\mu \ + \mathcal{L}_{MP}\ \delta x^\mu \,) \,]\ \mathrm{d}^d x \\
&=: \int_{\Gamma J^1\mathcal{E}^*} [\, E_{p_i^\mu}\delta p_i^\mu \ + \ E_{u^i}\delta u^i \ + \ \mathcal{H}_\mu \delta x^\mu \,]\ \mathrm{d}^d x \ + \ \int_{\partial\Gamma J^1\mathcal{E}^*} T_\delta^\mu \mathrm{d}S_\mu
\end{aligned} \tag{3.51}$$

where S_μ is the surface element of the spacetime boundary $\partial\Gamma J^1\mathcal{E}^*$ of the partial section $\Gamma J^1\mathcal{E}^*$ which is the region of integration. The coefficients are $E_{p_i^\mu} \ := \ \partial_\mu u^i - \frac{\partial \mathcal{H}}{\partial p_i^\mu} = \frac{\delta \mathcal{L}_{MP}}{\delta p_i^\mu}$ and $E_{u^i} \ = \ -\ (\partial_\mu p_i^\mu + \frac{\partial \mathcal{H}}{\partial u^i}) = \frac{\delta \mathcal{L}_{MP}}{\delta u^i}$. The current $T_\delta^\mu \ := \ \delta u^i p_i^\mu \ + \mathcal{L}_{MP}\ \delta x^\mu \ = \delta u^i p_i^\mu \ + (p_i^\kappa \partial_\kappa u^i - \mathcal{H})\delta x^\mu$ is the flow of action produced by the variation δ. $\mathcal{H}_\mu := \frac{\partial \mathcal{H}}{\partial x^\mu}$.

If $\delta S = 0$ for all infinitesimal variations δp_i^μ and δu^i (but $\delta x^\mu = 0$) around a given prolonged section, with $\delta u^i = 0$ on the boundary $\partial\Gamma J^1\mathcal{E}^*$, then it can be seen that $E_{p_i^\mu} = 0$ and $E_{u^i} = 0$, which are the the DeDonder Weyl equations (3.48), must hold. Solving the DDW equations for the multimomenta and substituting for the multimomenta in the integrand of the covariant multiphase-space action gives back the original Lagrangian density corresponding to $\mathcal{H}$: $\mathcal{L} = p_i^\mu \partial_\mu u^i - \mathcal{H}$ (which is the inverse Legendre transformation) as functions of (x, u^i, u_μ^i).

Similarly the integral of the multisymplectic d-form over a spacetime field configuration (a prolonged section $j^1\phi : B \longrightarrow \mathcal{E} \longrightarrow J^1\mathcal{E}$ on $\pi^{B,J^1\mathcal{E}}$) gives the action of the field:

$$\int_{j^1\phi} \widetilde{\Theta}_\mathcal{L} = \int_B j^1\phi^* \widetilde{\Theta}_\mathcal{L} = \int_B \mathcal{L}(j^1\phi) \mathrm{d}^d x = S[\phi] \tag{3.52}$$

and the solution of the variational problem

$$\delta_\xi \int_{j^1\phi} \widetilde{\Theta}_{\mathcal{L}} = \delta_\xi \int_B \mathcal{L}(j^1\phi)\mathrm{d}^d x = \delta_\xi S[\phi] = 0 \tag{3.53}$$

is given by

$$\delta_\xi \mathcal{L}(j^1\phi) = j^1\phi^*(\delta_\xi \widetilde{\Theta}_{\mathcal{L}}) = j^1\phi^*(\xi \lrcorner \mathrm{d}\widetilde{\Theta}_{\mathcal{L}}) = j^1\phi^*(\xi \lrcorner \widetilde{\Omega}_{\mathcal{L}}) = 0 \tag{3.54}$$

If we set $\xi = \left(\frac{\partial}{\partial u^i}\right)$ in $j^1\phi^*(\xi \lrcorner \mathrm{d}\widetilde{\Theta}_{\mathcal{L}}) = 0$ we obtain, in local coordinates, the Euler-Lagrange equations of motion of the field $\phi(x^\mu)$: $\frac{\partial}{\partial x^\mu}\left(\frac{\partial \mathcal{L}}{\partial u^i_\mu}\right) - \left(\frac{\partial \mathcal{L}}{\partial u^i}\right) = 0$, $u^i(x^\mu) = \phi^i(x^\mu)$, $u^i_\nu(x^\mu) := \frac{\partial \phi^i}{\partial x^\nu}(x^\mu)$.

An application of the multiphase space action in QFT is shown in section 3.5.

3.4.3 Multiphase-space action variation expressed using extended multi-Poisson brackets

The multiphase-space action variation above (3.51) can be expressed using extended multi-Poisson brackets (3.26) and (given the restriction on the form of $H = \mathcal{H} - p$ specified below in the use of the multiphase energy momentum tensor) $\widetilde{T}^\mu{}_\nu$ (3.66):

$$\begin{aligned}
\delta S_{MP} \;&=\; \delta \int_{\Gamma J^1\mathcal{E}^*, H=0} (p_i^\mu \partial_\mu u^i - p)\mathrm{d}^d x \;= \\
= \int_{\Gamma J^1\mathcal{E}^*} &[\,(\partial_\mu u^i - \frac{\partial \mathcal{H}}{\partial p_i^\mu})\delta p_i^\mu - (\partial_\mu p_i^\mu + \frac{\partial \mathcal{H}}{\partial u^i})\delta u^i + \frac{\partial \mathcal{H}}{\partial x^\mu}\delta x^\mu\,]\,\mathrm{d}^d x \;+\; \int_{\partial\Gamma J^1\mathcal{E}^*} (\,\delta u^i p_i^\mu + \mathcal{L}_{MP}\delta x^\mu\,)\mathrm{d}S_\mu \\
= \int_{\Gamma J^1\mathcal{E}^*} &[\,-\widetilde{\{u^i, H\}}_\mu \delta p_i^\mu + \widetilde{\{p_i^\mu, H\}}_\mu \delta u^i - \widetilde{\{p\,, H\}}_\mu \delta x^\mu\,]\,\mathrm{d}^d x \;+\; \int_{\partial\Gamma J^1\mathcal{E}^*} (\,\delta u^i p_i^\mu + \mathcal{L}_{MP}\delta x^\mu\,)\mathrm{d}S_\mu \\
= \int_{\Gamma J^1\mathcal{E}^*} &[\,-\widetilde{\{u^i\,,\, \widetilde{T}^\mu{}_\nu\,\}}_\mu \delta p_i^\nu + \widetilde{\{p_i^\nu\,,\, \widetilde{T}^\mu{}_\nu\,\}}_\mu \delta u^i - \widetilde{\{p\,, \widetilde{T}^\mu{}_\nu\,\}}_\mu \delta x^\nu\,]\,\mathrm{d}^d x + \int_{\partial\Gamma J^1\mathcal{E}^*} (\delta u^i p_i^\mu + \mathcal{L}_{MP}\delta x^\mu\,)\mathrm{d}S_\mu
\end{aligned} \tag{3.55}$$

where $\mathcal{L}_{MP} := p_i^\kappa \partial_\kappa u^i - \mathcal{H}$. The variational principle gives the DDW equations of motion (3.67) expressed with extended multi-Poisson brackets.

3.4.4 Energy momentum tensor

The above variation (3.51) is now expressed for the case of infinitesimal spacetime translation $\delta x^\nu(x)$:

$$\delta p_i^\mu = -\partial_\nu p_i^\mu \delta x^\nu \;,\; \delta u^i = -\partial_\nu u^i \delta x^\nu \;,\; \delta x^\nu \tag{3.56}$$

which gives

$$\begin{aligned}
\delta S_{MP} = \delta \int_{\Gamma J^1\mathcal{E}^*} (p_i^\mu \partial_\mu u^i - \mathcal{H})\mathrm{d}^d x = \\
= -\int_{\Gamma J^1\mathcal{E}^*} [\,(\partial_\mu u^i - \frac{\partial \mathcal{H}}{\partial p_i^\mu})\partial_\nu p_i^\mu \delta x^\nu \;-\; (\partial_\mu p_i^\mu + \frac{\partial \mathcal{H}}{\partial u^i})\partial_\nu u^i \delta x^\nu \;+\; \frac{\partial \mathcal{H}}{\partial x^\mu}\delta x^\mu
\end{aligned}$$

$$+\ \partial_\mu(\ \{\partial_\nu u^i p_i^\mu - (p_i^\kappa \partial_\kappa u^i - \mathcal{H})\ \}\ \delta x^\mu\ )\ ]\ \mathrm{d}^d x$$
$$= - \int_{\Gamma J^1 \mathcal{E}^*} [\ \{\ (\partial_\mu u^i - \frac{\partial \mathcal{H}}{\partial p_i^\mu})\partial_\nu p_i^\mu - (\partial_\mu p_i^\mu + \frac{\partial \mathcal{H}}{\partial u^i})\partial_\nu u^i + \frac{\partial \mathcal{H}}{\partial x^\nu}\ \}\ \delta x^\nu\ ]\ \mathrm{d}^d x$$
$$+ \int_{\partial \Gamma J^1 \mathcal{E}^*} (\{\ -\partial_\nu u^i p_i^\mu\ + (p_i^\kappa \partial_\kappa u^i - \mathcal{H})\delta_\nu^\mu\}\delta x^\nu\ )\ \ \mathrm{d}S_\mu$$
$$=\ -\ \int_{\Gamma J^1 \mathcal{E}^*} [\ \{\ (E_{p_i^\mu} \partial_\nu p_i^\mu\ +\ E_{u^i}\partial_\nu u^i\ \ -\mathcal{H}_\nu\ \}\ \delta x^\nu +\ \partial_\mu(T^\mu{}_\nu \delta x^\nu\ )\ ]\ \mathrm{d}^d x \tag{3.57}$$

The divergence which leads to the boundary term can be written as

$$\partial_\mu(T^\mu{}_\nu \delta x^\nu)\ =\ \partial_\mu((\partial_\nu u^i p_i^\mu\ - (p_i^\kappa \partial_\kappa u^i - \mathcal{H})\delta_\nu^\mu\ )\delta x^\nu) \tag{3.58}$$

where

$$T^\mu{}_\nu := \partial_\nu u^i p_i^\mu\ - (p_i^\kappa \partial_\kappa u^i - \mathcal{H})\delta_\nu^\mu = \partial_\nu u^i p_i^\mu\ - \mathcal{L}_{MP}\, \delta_\nu^\mu \tag{3.59}$$

is the energy-momentum tensor expressed with multimomenta as well as partial derivatives of the fields.

When the DDW equations hold and $\mathcal{H}$ is not explicitly a function of spacetime x^μ and if $\delta S_{MP} = 0$ for any globally constant translation, then $\partial_\mu(T^\mu{}_\nu \delta x^\nu)\ = \partial_\mu(T^\mu{}_\nu)\delta x^\nu\ =\ 0$ and so $T^\mu{}_\nu$ is a set of conserved currents.

Spacetime variations are used for studying certain spacetime symmetries such as the Poincare group of symmetries on Minkowski space. This was done in the subsection on the energy-momentum tensor in the section 2.3 on Lagrangian field theory.

We will now replace the partial derivatives of the fields using DDW1 (3.48): $\partial_\mu u^i(x) = \frac{\partial \mathcal{H}}{\partial p_i^\mu}(x, u^i(x), p_i^\mu(x))$. Thereby we obtain a covariant multiphase-space definition of the energy-momentum tensor density which does not involve derivatives of the fields, so that $T^\mu{}_\nu$ becomes a 'function' (observable) on covariant multiphase space. (More precisely, a spacetime tensor density on covariant multiphase space.)

Then, the multimomentum energy momentum tensor density is now defined to be:

$$\tilde{T^\mu_\nu}\ :=\ \frac{\partial \mathcal{H}}{\partial p_i^\nu} p_i^\mu\ - (p_i^\kappa \frac{\partial \mathcal{H}}{\partial p_i^\kappa} - \mathcal{H})\delta_\nu^\mu\ =\ \mathcal{U}_\nu^i p_i^\mu\ - \tilde{\mathcal{L}}_{MP}\ \delta_\nu^\mu\ =\ \mathcal{U}_{\nu'}^i p_i^{\mu'}\ \delta_{\mu'\nu}^{\mu\ \nu'}\ +\ \mathcal{H}\ \delta_\nu^\mu \tag{3.60}$$

where the following notation is used: the strain observable $\mathcal{U}_\nu^i := \frac{\partial \mathcal{H}}{\partial p_i^\nu}$, the multiphase-space Lagrangian observable $\tilde{\mathcal{L}}_{MP} := p_i^\kappa \frac{\partial \mathcal{H}}{\partial p_i^\kappa} - \mathcal{H}$, and the antisymmetrizer $\delta_{\mu'\nu}^{\mu\ \nu'} := \delta_{\mu'}^\mu \delta_\nu^{\nu'} - \delta_\nu^\mu \delta_{\mu'}^{\nu'}$.

Calculating the multi-Poisson bracket (defined in section 3.2) of a general function $\mathcal{O}$ on multiphase space with the energy momentum tensor:

$$\{\mathcal{O}\ ,\ \tilde{T^\mu}_\nu \mathrm{d}x_\mu\} = \{\mathcal{O}\ ,\ \mathcal{H}\}_\nu\ +\ 2\ [\frac{\partial \mathcal{O}}{\partial u^i} \frac{\partial \mathcal{H}}{\partial p_i^{[\kappa} \partial p_j^{\nu]}} p_j^\kappa\ -\ \frac{\partial \mathcal{O}}{\partial p_i^{[\kappa}} \frac{\partial \mathcal{H}}{\partial p_j^{\nu]} \partial u^i} p_j^\kappa\ ] \tag{3.61}$$

$$\{\mathcal{O}\ ,\ \tilde{T^{\mu}}_{\nu}\mathrm{d}x_{\mu}\} = \{\mathcal{O}\ ,\ \mathcal{H}\}_{\nu}\ +\ 2p_j^{\kappa}\Big(\frac{\partial\mathcal{O}}{\partial u^i}\frac{\partial\mathcal{H}}{\partial p_i^{[\kappa}\partial p_j^{\nu]}}\ -\ \frac{\partial\mathcal{O}}{\partial p_i^{[\kappa}}\frac{\partial\mathcal{H}}{\partial p_j^{\nu]}\partial u^i}\Big) \tag{3.62}$$

$$\{\mathcal{O}\ ,\ \tilde{T^{\mu}}_{\nu}\mathrm{d}x_{\mu}\} = \{\mathcal{O}\ ,\ \mathcal{H}\}_{\nu}\ +\ 2p_j^{\kappa}\{\mathcal{O}\ ,\ \frac{\partial\mathcal{H}}{\partial p_j^{[\nu}}\}_{\kappa]} \tag{3.63}$$

For $\mathcal{O} = u^i$ and $\mathcal{O} = p_i^{\nu}$ we obtain

$$\{u^j, \tilde{T^{\mu}}_{\nu}\mathrm{d}x_{\mu}\} = \frac{\partial\mathcal{H}}{\partial p_j^{\nu}}\ +\ 2p_k^{\kappa}\Big(\frac{\partial\mathcal{H}}{\partial p_j^{[\kappa}\partial p_k^{\nu]}}\Big)\quad ,\quad \{p_i^{\nu}, \tilde{T^{\mu}}_{\nu}\}_{\mu} =\ -\ \frac{\partial\mathcal{H}}{\partial u^i}\ \approx \partial_{\nu}p_i^{\nu} \tag{3.64}$$

The above term $2p_k^{\kappa}\Big(\frac{\partial\mathcal{H}}{\partial p_j^{[\kappa}\partial p_k^{\nu]}}\Big)$ is zero if $\mathcal{H}$ is of the form $\mathcal{H} = a(u,x) + b_{\alpha}^{a}(u,x)p_a^{\alpha} + g_{\alpha\beta\gamma\ldots}^{abc\ldots}(u,x)p_a^{\alpha}p_b^{\beta}p_c^{\gamma}\ldots$, where $g_{\alpha\beta\gamma\ldots}^{abc\ldots}(u)$ is symmetric for all exchanges of up-down pairs of indices, such as $g_{\alpha\beta\gamma\ldots}^{abc\ldots} = g_{\alpha\gamma\beta\ldots}^{acb\ldots}$ etc. In that case, we have as desired

$$\{u^j, \tilde{T^{\mu}}_{\nu}\mathrm{d}x_{\mu}\} = \frac{\partial\mathcal{H}}{\partial p_j^{\nu}}\ \approx \partial_{\nu}u^j. \tag{3.65}$$

This shows that the multiphase-space energy momentum tensor $\tilde{T}$ generates the DDW equations of motion, via the multi-Poisson bracket. In one dimensional spacetime, where the multiphase-space mechanics reduces to Hamiltonian mechanics, the energy momentum tensor above is the Hamiltonian function, and the multi-Poisson brackets are the Poisson brackets with which the Hamiltonian generate Hamilton's equations. So the multiphase-space energy momentum tensor originating from the equation $\partial_{\mu}T^{\mu}{}_{\nu} + \frac{\partial\mathcal{H}}{\partial x^{\nu}} = \frac{\delta S_{MP}}{\delta x^{\nu}}$ can be viewed as a generalization of the Hamiltonian $\mathcal{H} = \frac{\delta S_P}{\delta t}\Big|_{q,p}$ of Hamiltonian mechanics.

Using the multiphase-space energy momentum tensor $\tilde{T}^{\mu}{}_{\nu}$ above defined on *covariant multiphase space*, on *multiphase space* we can define

$$\widetilde{T}^{\mu}{}_{\nu} := \frac{\partial\mathcal{H}}{\partial p_i^{\nu}}p_i^{\mu}\ -(p_i^{\kappa}\frac{\partial\mathcal{H}}{\partial p_i^{\kappa}} - \mathcal{H})\delta_{\nu}^{\mu}\ =\ \mathcal{U}_{\nu'}^{i}p_i^{\mu'}\ \delta_{\nu\mu'}^{\mu\nu'}\ +\ \mathcal{H}\ \delta_{\nu}^{\mu} \tag{3.66}$$

which can be used to write the DeDonderWeyl equations of motion in the form

$$0\ \approx\ \widetilde{\{}u^i\ ,\ \widetilde{T}^{\mu}{}_{\nu}\mathrm{d}x_{\mu}\}\quad\text{and}\quad 0\ \approx\ \widetilde{\{}p_i^{\nu}\ ,\ \widetilde{T}^{\mu}{}_{\nu}\mathrm{d}x_{\mu}\} \tag{3.67}$$

The energy momentum tensor appears as the generator of spacetime translation in the next section.

3.5 Application of multiphase space action in QFT

In this section we examine the functional integral of the complex exponential of the multiphase space action, relating it to the Feynman functional integral in QFT, and conjecture a notion of quantum operators insertions in spacetime.

The Dirac canonical quantization of fields is based on the phase space Hamiltonian formalism in classical mechanics and has the same features - in that it singles out time from space and is not inherently local - whereas multiphase space is inherently covariant and local in spacetime. The Feynman functional integral based on a Lagrangean density from classical field theory can be manifestly covariant, but has the disadvantage that unitarity and the Hilbert space structure is not manifest.

In particular, the well known Feynman configuration space path integral can be obtained from the path integral of fields with a multiphase space Lagrangian density, and this can be employed to investigate the meaning of multimomentum operators and multi-brackets in quantum field theory, as sketched below. It then may be possible to develop a covariant QFT operator algebra on manifolds, as a generalization of the Lie algebra of operators in the Heisenberg picture of QM. The algebra has a natural d-dimensional manifold structure, arising from the fact that an infinitesimal multi-evolution operator sits at each point of spacetime and 'multiplies' nearest neighbours, as a generalization of the time-sequence of products of the time-sliced evolution operator. These could be defined using discretized functional integrals on multiphase space.

In this section we will limit ourselves to describing, without giving the precise derivation, how path integrals on multiphase space are equal to the usual configuration space path integrals, in the case that the DDW Hamiltonian in the multiphase space Lagrangian density is quadratic and non-degenerate in the multimomenta.

We start with a reminder of the straightforward construction of the Feynman quantum path integral for particles starting from the canonical operator formalism. This leads, in the first instance, to a path integral over all paths in phase space of the complex exponential of the phase space action, where the Hamiltonian of the system appears in the phase space Lagrangian. When the phase space Lagrangian is of a suitable form, quadratic and non-degenerate in the momenta in the simplest case, the momenta can be integrated out (as a gaussian integral), leaving the Feynman path integral on configuration space of the complex exponential of the usual configuration space action. This is the usual Feynman path integral. We can use the same construction on fields, by firstly discretizing the spatial coordinates of the field, and considering the discrete spatial parameter as similar to labelling different particles. We then go over to the continuum limit on the integrals, making the discretization denser and obtain in the limit the functional integral of the complex exponential of the action of the field configurations. Starting from the latter, one could consider reversing this process to constuct a path integral with auxiliary variables, the momenta, to eliminate the time derivative instead, thereby recovering the functional integral of the complex exponential of the phase space Lagrangian. This, of course, is the same phase space path integral above constructed from the canonical formalism. Now, the point here is, one could also consider continuing this process by eliminating the spatial derivatives of the field as we did the time derivatives. This is done the same way as was done to the time derivative - by introducing extra auxiliary variables - and we would, in fact, thereby obtain the functional integral of the complex exponential of the multiphase space action, with the DDW Hamiltonian of the classical fields, and the extra variables are multimomenta. This is explained in a little more detail in the next paragraph.

In a little more detail we expand the sketch in the previous paragraph. We start with summarizing how the usual Feynman path integral is constructed from the canonical Hamiltonian formalism: In canonical QM, the unitary evolution operator $U(T) = \exp \frac{i}{\hbar} HT$, where H is the quantum Hamiltonian operator, which acts to change the state at time 0 to the time-evolved state at time T is factorized into the product of a large number, $N = T/\delta t$, of infinitesimal-time evolution operators: $\Pi_1^N U(\delta t) = \Pi_1^N (1 + \frac{i}{\hbar} \delta t H)$. For a state $\Psi(t)$, this is the Schrodinger equation $\frac{\mathrm{d}\Psi}{\mathrm{d}t} = \frac{i}{\hbar} H(\hat{q}, \hat{p}) \Psi(t)$. If one were to discretize the factors $U(\delta t)$ as K by K matrices, then the entire calculation could be performed as a sum of K^N products, each product having consisting of N matrix elements as factors, one from each matrix in the sequence. Each of these products is a path $\mathcal{P}$ through the sequence of matrices from discretized time $i = 0$ to time $i = N$, where the position of this path at time i is $k_i(\mathcal{P}) \in \{1, \ldots, K\}$ and the corresponding ith factor of the product at this point of the path $\mathcal{P}$ is the matrix element $U(\delta t)_{k_i k_{i+1}}$. In the continuum limit, this is the starting point of the path integral fomulation of QM, and each term (path) in the sum turns out to be the complex exponential of the classical phase space action for that path: $e^{iS_P[\mathcal{P}]/\hbar}$ (ignoring issues to do with the measure when the sum over paths is now a functional integral). We now on consider the system to be a field with an infinite number of degrees of freedom parametrized by the spatial coordinates: If one were to integrate out (before going to the continuum limit) the momentum variables for a simple quadratic dependance of the Hamiltonian on the momentum, the contribution of each path is now the complex exponential of the classical action for each path: $e^{iS[\mathcal{P}]/\hbar}$ (again ignoring issues to do with the measure, and constant factors). The main effect of integrating out the momenta is to replace the $\int p_a \partial_t \phi^a - \frac{1}{2g} |p_a(x)|^2 \mathrm{d}x^{d-1}$ term in the phase space Lagrangian with $\int \frac{g}{2} |\partial_t \phi^a|^2 \mathrm{d}x^{d-1}$. If, instead of integrating out the momenta, we do the reverse process, as suggested in the previous paragraph, and convert the squared spatial derivatives $\frac{1}{2} \partial_i \phi^a \partial^i \phi_a$ in the Lagrangian density to $\frac{1}{2} p_a^i p_i^a$ with extra auxiliary variables p_a^i we would, in fact, obtain a path integral $\int D[u^i(x), p_i^\mu(x)] \; e^{\frac{i}{\hbar} S_{MP}[u^i(x), p_i^\mu(x)]}$ of complex exponentials of the multi-phase space action.

This suggests that the multiphase space formalism is a natural one for quantum functional integrals of fields, in some analogous way that, for particles, the phase space path integral can be naturally constructed from the canonical operator formalism. It is therefore interesting to investigate whether there might be an analogue of the phase space canonical operator formalism using multiphase space.

Local operators in spacetime

In canonical QM, operator ordering amounts to time ordering of insertions of operators in the path integral formulation of QM. This can be generalized to functional integration of fields in covariant QFT, where spacetime ordering of insertions of observables at the same spacetime points corresponds to operator ordering with a directional parameter μ.

We can now define a local operator $\hat{\mathcal{O}} = \mathcal{O}(\hat{q}^i_p, \hat{p}^\mu_{ip})$ at the point $p = (txyz)$ in Minkowski space via matrix elements calculated from this functional integral in the same way as ordinary QFT:

$$< \mathbf{q}(0)|\hat{\mathcal{O}}(txyz)|\mathbf{q}(T) > = \int_{\mathbf{q}(0) \text{ all paths } \mathcal{P} \text{ in MPS}}^{\mathbf{q}(T)} D[\mathcal{P}]\ \mathcal{O}(q(txyz), p^\mu(txyz))\ e^{iS_{MP}[\mathcal{P}]/\hbar} \tag{3.68}$$

$\mathcal{O}$ is a function of the fields $q(txyz)$ and of the multimomenta $p^\mu(txyz)$ and their spacetime derivatives. The difference with ordinary QFT is that in the latter one starts with local operators and shows that inserting time-ordered operators is equal to inserting the corresponding classical observables in the path integral on the right hand side. In the multimomentum setting we can use the path integral to define operators and to obtain operator identities. In particular, the multi-Posson brackets of observables correspond to directional commutator brackets of observables.

This definition can be generalized to non-local operators in the same way that in ordinary QFT insertions into path integrals correspond to time-ordered operators $T[\mathcal{O}]$, so here the spacetime ordering of operators is indicated by the $S[]$ notation:

$$< S[\hat{\mathcal{O}}] > = \int_{\mathbf{q}(t_i) \text{ all paths } \mathcal{P} \text{ in MPS}}^{\mathbf{q}(t_f)} D[\mathcal{P}]\ \mathcal{O}\ e^{iS_{MP}[\mathcal{P}]/\hbar} \tag{3.69}$$

Examples of observables of interest to be inserted into the functional integrals and be promoted to operators are the fields q^i, the multimomenta p^μ_i, the DDW Hamiltonian $\mathcal{H}$ the energy momentum tensor T^μ_ν, multi-brackets of observables, and the DDW equations of motion. It turns out that the results are what would be expected if the formalism works: the multi-bracket result is $\{q^i, p^\mu\}_\nu = \delta^\mu_\nu$, the DDW equations of motion hold, the multi-Poisson bracket $i\hbar\{A, B\}_\nu$ has the same effect as inserting the operator A immediately to the left (decreasing coordinate in the direction ν) to operator B in the lattice and subtracting the same thing with A and B reversed: $i\hbar\{A, B\}_\nu \longrightarrow [\hat{A}, \hat{B}]_\nu$. Spacetime translation is obtained by $\{\mathcal{O}, T^\mu_\nu\}_\mu = \partial_\nu\mathcal{O}$, so the energy momentum tensor generates translation.

3.6 Symmetries and constraints in multiphase space

This section is analogous to section 2.4 'Symmetries and constraints', but generalized to multiphase space. We will examine the theory of fields with primary and secondary constraints which is necessary for Yang-Mills fields. Then we will look at the abstract theory of hamiltonian multisymplectomorphism group actions on multisymplectic manifolds and Marsden-Weinstein reduction. After short summaries of examples of mutisymplectic field theories in the appendix we develop at length the example of multisymplectic non-abelian Yang-Mills field theory.

3.6.1 Symmetry: multiphase-space Lagrangian

A variation δ in a path in a configuration action can be extended to the multiphase-space action, by ensuring that the variation of the multimomenta is consistent with the equations of motion (the Euler-Lagrange equations for the multiphase-space Lagrangian density which are also the DDW equations).

We assume the set of symmetries on the multiphase-space action form a group. The group may be discrete - such as time reversal and space inversion symmetries - but we will concern ourselves here with continuous symmetries forming Lie groups. The corresponding vector space of infinitesimal symmetries around the identity will form a Lie algebra. From the Lie algebra it is possible to reconstruct the part of the Lie group connected to the identity, therefore in studying symmetry it is often only necessary to deal with the Lie algebra and its (infinitesimal) action rather than the full group of symmetries. Often it is convenient to consider one dimensional subgroups of the Lie group of symmetries, i.e. a one parameter group.

If the multiphase-space action is invariant under some continuous group of symmetries then it is necessary to distinguish between two different situations (1) global symmetries and (2) local symmetries.

(1) If the symmetry is a continuous one parameter group of symmetries, then we first consider the case of a global symmetry where each path is mapped into another path, with different starting values. To this kind of symmetry is associated a spacetime $d-1$-form multiphase-space observable which is a conserved current on trajectories, that is, evolutions (paths) which obey the DDW equations of motion. This is Noether's theorem on multiphase space. Specifically, if the infinitesimal symmetry variation has $\delta S_{MP}[q^i(x), p_i^\mu(x)] = 0$ then equation (3.51) shows that, when the Euler-Lagrange equations are satisfied,

$$0 = \delta S_{MP}[q^i(x), p_i^\mu(x)] \;= \int_{\Gamma J^1\mathcal{E}^*} [\; \frac{\partial \mathcal{H}}{\partial x^\mu}\delta x^\mu \;+ \partial_\mu(\, \delta u^i p_i^\mu \;+ (p_i^\kappa \partial_\kappa u^i - \mathcal{H})\delta x^\mu \,)\;]\, \mathrm{d}^d x$$

$$= \int_{\Gamma J^1\mathcal{E}^*} \frac{\partial \mathcal{H}}{\partial x^\mu}\delta x^\mu \; \mathrm{d}^d x \; + \int_{\partial\Gamma J^1\mathcal{E}^*} (\, \delta u^i p_i^\mu \; + (p_i^\kappa \partial_\kappa u^i - \mathcal{H})\delta x^\mu \,) \; \mathrm{d}S_\mu \qquad (3.70)$$

When the DDW Hamiltonian density $\mathcal{H}$ is not explicitly a function of spacetime coordinates, x^μ, and the variation leaves the multiphase-space action invariant, the above shows that $J^\mu := \delta u^i p_i^\mu \; + (p_i^\kappa \partial_\kappa u^i - \mathcal{H})\delta x^\mu$ is a conserved current, $\partial_\mu J^\mu = 0$, on a trajectory. This situation often arises from physical symmetries such as space-translation symmetry where the energy-momentum tensor density (3.60) is the corresponding conserved current. When there is a symmetry group, we will have a set of conserved currents indexed by a basis of the Lie algebra of the symmetry: J_a^μ and a particular global infinitesimal variation with parameters (f^a) will be associated with a current $f^a J_a^\mu$.

(2) A local gauge symmetry is a continuous group of symmetries where a trajectory can be varied locally and independently at each point in spacetime with the action remaining unchanged. In this case, the fields at the spacetime boundary can be kept fixed, but, by hypothesis, the configuration can be changed continuously in the interior without changing the action. This is a situation where the action functional does not have a unique distinct stationary point (trajectory), for given boundary values. In this case the Euler-Lagrange equations fro the fields will be degenerate and the action principle is not sufficient to specify one trajectory, and thus there will therefore be a class of trajectories which satisfy the action principle for given fixed boundary values. One may view this as an incompletely specified field model, or alternatively, a model where more degrees of freedom are used to specify the system than are strictly necessary to specify the physical configuration. The extra non-physical degrees of freedom is the gauge freedom to vary observables and trajectories, without changing the gauge class of that trajectory, each class representing one physical trajectory. A well known example are covariant general relativistic actions where there is the gauge freedom of varying the underlying spacetime coordinate system - embodying a geometrical principle of general coordinate invariance. In these cases there are gauge degrees of freedom which are not physical.

The abstract relationship between the original multiphase space with symmetry and the physical reduced multiphase space in terms of multisymplectic manifolds is a generalization of as Marsden-Weinstein reduction [6] described in the section (3.6.3).

3.6.2 Field theory with primary and secondary constraints

This section will be used in the multiphase-space BRST technique later for Yang-Mills fields which have primary and secondary constraints. The BRST model is constructed from the constraint algebra, so we are interested in whether the multibrackets have the requisite properties to function as a Poisson algebra in BRST. In fact, as we shall see here, the very simple structure of the observables involved allow the multibrackets to give the right algebra.

Here we examine a system with primary and secondary, but not tertiary, constraints. To start with some conclusions of this section: 1) secondary constraints are not truly constraints in that they are not fixed in the same way primary constraints are, 2) secondary constraints are conserved (Noether) currents.

We start with a field action which is invariant under the infinitesimal gauge variation of the fields $u^i(x)$ given by the parameters $\epsilon^a(x)$ and the (first) spacetime derivatives $\partial_\mu\epsilon^a(x)$:

$\delta_\epsilon(u^i(x)) = \partial_\mu\epsilon^a(x)S_a^{i\mu}(u(x)) + \epsilon^a(x)R_a^i(u(x))$ where $S_a^{i\mu}(u), R_a^i(u)$ are functions specifying the gauge variations. Then the variation of the action is:

$$0 = \delta S = \delta \int \mathcal{L}(u(x), u_\mu(x), x)\ \mathrm{d}^d x =$$

$$\begin{aligned}\int\ [(\ \partial_\mu\epsilon^a(x)S_a^{i\mu}(u(x)) + \epsilon^a(x)R_a^i(u(x))\ )\ L_i(u(x), u_\mu(x), x)\\ +\ (\ \partial_\alpha\partial_\mu\epsilon^a(x)S_a^{i\mu}(u(x)) + \partial_\mu\epsilon^a(x)\partial_\alpha S_a^{i\mu}(u(x))\\ +\ \partial_\alpha\epsilon^a(x)R_a^i(u(x)) + \epsilon^a(x)\partial_\alpha R_a^i(u(x))\ )\ P_i^\alpha(u(x), u_\mu(x), x)\ ]\ \mathrm{d}^d x\end{aligned} \tag{3.71}$$

where $L_i := \frac{\partial\mathcal{L}}{\partial u^i}$ and $P_i^\alpha := \frac{\partial\mathcal{L}}{\partial u_\alpha^i}$

Because we impose the condition that the action is invariant for arbitrary functions $\epsilon^a(x)$, the coefficients of the zeroth, first and second derivatives above must be pointwise constant:

$$R_a^i\ L_i + \partial_\alpha R_a^i\ P_i^\alpha = 0 \tag{3.72}$$

$$S_a^{i\mu}\ L_i + \partial_\alpha S_a^{i\mu}\ P_i^\alpha + R_a^i\ P_i^\mu = 0 \tag{3.73}$$

$$S_a^{i(\mu}P_i^{\alpha)} = 0 \tag{3.74}$$

From the first equation, $\partial_\mu(R_a^i P_i^\mu) \approx 0$ is true when the Euler-Lagrange equations hold. $J_a^\mu := R_a^i P_i^\mu$ are the conserved Noether currents. The third equation (3.74), when the Legendre transformation is performed, shows that there are constraints on multiphase space $T_a^{(\mu\alpha)} = S_a^{i(\mu}p_i^{\alpha)} = 0$. We define the primary generator $T_a^{(1)\mu\alpha} = S_a^{i\mu}p_i^\alpha$ and the secondary generator $T_a^{(2)\alpha} = R_a^i p_i^\alpha$. The second equation (3.73) shows that the primary generator is not conserved: $\partial_\alpha T_a^{(1)\mu\alpha} = T_a^{(2)\mu}$.

Because the T's are linear in the multimomenta it can be seen (as in section D.1) that this can be expressed with multibrackets. The conservation of the Noether current (secondary generator)

$$\partial_\nu \mathrm{T}_a^{(2)\nu} \approx \{T_a^{(2)\alpha}\mathrm{d}x_\alpha, \mathcal{H}\}\ =\ 0 \tag{3.75}$$

and the non conservation of the primary generator:

$$\partial_\nu \mathrm{T}_a^{(1)\alpha\nu} \approx \{T_a^{(1)\alpha\nu}\mathrm{d}x_\nu, \mathcal{H}\}\ =\ T_a^{(2)\alpha} \tag{3.76}$$

The form of the gauge generator, for the infinitesimal gauge variation δ_ϵ, in a system with primary and secondary constraints is :

$$T_\epsilon = \partial_\nu \epsilon^a(x) T_a^{(1)\nu} + \epsilon^a T_a^{(2)} = \partial_\nu \epsilon^a(x) T_a^{(1)\nu\mu} \mathrm{d}x_\mu + \epsilon^a T_a^{(2)\mu} \mathrm{d}x_\mu$$
$$= \partial_\nu \epsilon^a(x) S_a^{i\nu} p_i^\mu \mathrm{d}x_\mu + \epsilon^a R_a^i p_i^\mu \mathrm{d}x_\mu \tag{3.77}$$

where $T_a^{(1)\nu}$ are the primary constraint and $T_b^{(2)}$ are the secondary constraint $d-1$ -forms.

This can be seen by explicitly calculating the variation using the generator above (Note that we are only requiring the multibracket to have 'nice' properties when acting on observables of the form $f^i(u(x))\bar{p}_i = f^i(u(x))p_i^\mu \mathrm{d}x_\mu$) :

$$\delta_\epsilon(f^i(u(x))\bar{p}_i) = \{T_\epsilon, f^i(u(x))\bar{p}_i\} = \{\partial_\nu\epsilon^a(x)S_a^{i\nu}\bar{p}_i + \epsilon^a R_a^i \bar{p}_i, f^i(u(x))\bar{p}_i\}$$
$$= \partial_\nu\epsilon^a(x)\{S_a^{i\nu}\bar{p}_i, f^i(u(x))\bar{p}_i\} + \epsilon^a\{R_a^i\bar{p}_i, f^i(u(x))\bar{p}_i\}$$
$$= \partial_\nu\epsilon^a(x)(\bar{p}_i\{S_a^{i\nu}, \bar{p}_j\}f^j(u(x)) + S_a^{i\nu}\{\bar{p}_i, f^j(u(x))\})\bar{p}_j$$
$$+\epsilon^a(\bar{p}_i\{R_a^i, \bar{p}_j\}f^j(u(x) + R_a^i\{\bar{p}_i, f^j(u(x))\}\bar{p}_j)$$
$$= \partial_\nu\epsilon^a(\bar{p}_i S_{a,j}^{i\nu} f^j - S_a^{i\nu} f^j_{,i}\bar{p}_j) + \epsilon^a(\bar{p}_i R_{a,j}^i f^j - R_a^i f^j_{,i}\bar{p}_j)$$
$$= [\partial_\nu\epsilon^a(S_{a,j}^{i\nu} f^j - S_a^{j\nu} f^i_{,j}) + \epsilon^a(R_{a,j}^i f^j - R_a^j f^i_{,j})]\bar{p}_i \tag{3.78}$$

where $f_{,i} := df/du^i$. From this we obtain

$$\delta_\epsilon \bar{p}_j = [\partial_\nu \epsilon^a S_{a,j}^{i\nu} + \epsilon^a R_{a,j}^i]\bar{p}_i \tag{3.79}$$

$$\delta_\epsilon u^i = \partial_\nu \epsilon^a(x) S_a^{i\nu} + \epsilon^a R_a^i \tag{3.80}$$

$$\delta_\epsilon f(u) = \partial_\nu \epsilon^a(x) S_a^{i\nu} f_{,i} + \epsilon^a R_a^i f_{,i} \tag{3.81}$$

Because $T_a^{(1)\nu}$ and $T_a^{(2)}$ are of the simple form $T_a^{(1)\nu\mu}\mathrm{d}x_\mu = S_a^{i\nu}p_i^\mu \mathrm{d}x_\mu$ and $T_a^{(2)\mu}\mathrm{d}x_\mu = R_a^i p_i^\mu \mathrm{d}x_\mu$ which is linear in the multimomenta, the multibrackets act like Lie brackets if the gauge transformations act as a Lie algebra, with structure constants f_{ab}^c, pointwise on spacetime: $\epsilon_3^c(x) = f_{ab}^c \epsilon_1^a(x)\epsilon_2^b(x)$ where the corresponding action is $\delta_{\epsilon_3} = [\delta_{\epsilon_2}, \delta_{\epsilon_1}]$.

The fact that there is a Lie algebra map between $(\delta_\epsilon, [,])$ and $(T^{(2)}, \{,\})$ is shown in section D.1, in the context of a global symmetry. This can be applied here because we are assuming that the structure constants do not vary.

Collecting together the brackets we have

$$\{T_a^{(1)\alpha}, \mathcal{H}\} \;=\; T_a^{(2)\alpha} \tag{3.82}$$

$$\{T_a^{(2)}, \mathcal{H}\} \;=\; 0 \tag{3.83}$$

If the generators have the same bracket algebra as the Lie algebra then the following must hold:

$$\{T_a^{(1)\alpha}, T_b^{(1)\beta}\} \;=\; 0 \tag{3.84}$$

$$\{T_a^{(1)\alpha}, T_b^{(2)}\} \;=\; -f_{ab}^c\, T_c^{(1)\alpha} \tag{3.85}$$

$$\{T_a^{(2)}, T_b^{(2)}\} \;=\; f_{ab}^c T_c^{(2)} \tag{3.86}$$

The constraint algebra above will be invaluable for constructing the BRST observable for Yang-Mills later in the next chapter.

3.6.3 Hamiltonian constraints and Marsden-Weinstein multisymplectic reduction

Lie group action on a multisymplectic manifold

We will consider a certain class of symmetries: diffeomorphisms of a multisymplectic manifold, which are also multisymplectomorphisms or exact multisymplectomorphisms. These will be the action of a Lie group G, on the multisymplectic manifold $\mathbb{M}$, which preserve the multisymplectic form. These which will lead to a constraint submanifold $\mathbb{M}_G \subset \mathbb{M}$ which is a premultisymplectic submanifold of $\mathbb{M}$. This submanifold is in turn foliated by the orbits of the symmetry transformation. Under the conditions that the action is free and proper, the space of orbits (displaying various notations seen in the literature), $\mathbb{M}_G/G = (\mathbb{M}_G)^G =: \mathbb{M}//G =: \tilde{\mathbb{M}} =: \mathbb{M}_{GG}$, is a manifold or an orbifold with a multisymplectic form inherited, via the embedding, from the original multisymplectic manifold, and is denoted the reduced multiphase space. This reduced multiphase space may be the physical multiphase space, whereas the original multiphase space has non-physical gauge or symmetry degrees of freedom, which it may be necessary or useful to retain for part of the study of the system, but from which we finally want to obtain the dynamics on the reduced multiphase space. We will deal with multisymplectomorphisms which are also hamiltonian - that is, the action infinitesimal generators of G are infinitesimal flows on $\mathbb{M}$ which are hamiltonian vector fields (whose hamiltonian 'functions' are $d-1$-forms). We also want to have generalized hamiltonian vector fields, whose hamiltonian 'functions' are more than simple $d-1$-forms. We will also deal with multisymplectomorphisms which are Poisson as well as hamiltonian - that is, the Lie algebra of the infinitesimal generators of G map to the multiPoisson algebra of the hamiltonian $d-1$-forms corresponding to the infinitesimal flows on $\mathbb{M}$ of a hamiltonian action.

Hamiltonian action

First the notion of hamiltonian multisymplectomorphism is defined.

The equation $X_F \lrcorner \Omega = \mathrm{d}F$, where F is a given spacetime $d-1$-form on a multisymplectic manifold $(\mathbb{M}, \Omega)$, if it has a solution X_F, the solution will a vector field $X_F = \{\cdot, F\}$, and this vector field is unique because Ω is non-degenerate by definition. The vector field X_F on $\mathbb{M}$ is denoted the hamiltonian vector field generated by the hamiltonian $d-1$-form F. This vector field, as an infinitesimal transformation of the manifold, is an infinitesimal multisymplectomorphism of $(\mathbb{M}, \Omega)$ because: $\mathcal{L}_{X_F}\Omega = X_F \lrcorner \mathrm{d}\Omega + \mathrm{d}(X_F \lrcorner \Omega) = X_F \lrcorner 0 + \mathrm{dd}F = 0$ (because $\mathrm{d}\Omega = 0$ by definition and $\mathrm{dd} = 0$, the property of the exterior derivative). Such a multisymplectomorphism is called a hamiltonian multisymplectomorphism.

Conversely, any vector field X_F which satisfies the equation $X_F \lrcorner \Omega = \mathrm{d}F$ for some F is called a hamiltonian vector field and is a generator of a one dimensional Lie group of global hamiltonian multisymplectomorphisms of $\mathbb{M}$, the flow of the vector field X_F. F is called the hamiltonian $d-1$-form or observable corresponding to the hamiltonian vector field X_F. The hamiltonian $d-1$-form F corresponding to the hamiltonian vector field X_F can have an arbitrary closed form added to it, without changing the relation $X_F \lrcorner \Omega = \mathrm{d}F$, because the relation involves $\mathrm{d}F$ rather than F.

Note that a arbitrary multisymplectomorphism Y has $0 = \mathcal{L}_Y\Omega = Y \lrcorner \mathrm{d}\Omega + \mathrm{d}(Y \lrcorner \Omega) = \mathrm{d}(Y \lrcorner \Omega) =: \mathrm{d}\omega_Y$ so $\omega_Y := Y \lrcorner \Omega$ is closed d-form. By the Poincare lemma, for any d-form ω on any contractible open submanifold of $\mathbb{M}$, there will exist a $d-1$-form h such that $\mathrm{d}h = \omega$. In our case this means that there exists locally a hamiltonian $d-1$-form h_Y such that $\mathrm{d}h_Y = Y \lrcorner \Omega$. In the case of an exact multisymplectomorphism, $\mathcal{L}_Y\Theta = 0$, where $\Omega = -\mathrm{d}\Theta$, described in the examples below, the hamiltonian form is constructed directly from the hamiltonian vector field as $h_Y = Y \lrcorner \Theta$.

If the d-form $X_\xi \lrcorner \Omega = \xi$ is closed rather than exact, the vector field X_ξ is called a locally hamiltonian vector field and is a generator of a one dimensional Lie group of local multisymplectomorphisms of $\mathbb{M}$. ξ is called the hamiltonian d-form corresponding to that locally hamiltonian vector field. On a contractible open patch a locally hamiltonian vector field is hamiltonian. If $\mathfrak{g}$ is a Lie algebra acting on a multisymplectic manifold by such infinitesimal hamiltonian multisymplectomorphisms, then there is a hamiltonian vectorfield $X_{\eta_\mathbb{M}}$ and observable (called a constraint in this context) h_η corresponding to each element $\eta \in \mathfrak{g}$. Such an action is called a hamiltonian action h of the Lie algebra and there is a hamiltonian map from the Lie algebra to the space of spacetime $d-1$-forms on $\mathbb{M}$.

Multi-Poisson brackets

The hamiltonian map from the Lie algebra of a hamiltonian action to the space of spacetime $d-1$-forms on $\mathbb{M}$ is a linear map over $\mathbb{R}$ and suggests that there is a Lie algebra homomorphism from the Lie algebra of the symmetry group, the Lie algebra of the symmetry vector fields and some Lie algebra of the corresponding hamiltonian $d-1$-forms, where the hamiltonian $d-1$-form $h_{[X_a,X_b]}$, corresponding to the Lie bracket $[X_a, X_b]$ of the symmetry vector fields X_a, X_b, (i.e $\mathrm{d}h_{[X_a,X_b]} = [X_a, X_b] \lrcorner \Omega$), is the result of a bracket operation between the two hamiltonian $d-1$-forms h_{X_a} and h_{X_b}:

$$h_{[X_a,X_b]} = \{h_{X_a}, h_{X_b}\} \tag{3.87}$$

The bracket is defined via the corresponding vector fields X_a, X_b:

$$\{h_{X_a}, h_{X_b}\} := X_a \lrcorner \mathrm{d}h_{X_b} = X_a \lrcorner X_b \lrcorner \Omega \tag{3.88}$$

This bracket is clearly bilinear and antisymmetric from the definition.

In fact in the case of exact multisymplectomorphisms, $\mathcal{L}_{X_F}\Theta = 0$, where $\Omega = -\mathrm{d}\Theta$ (see below in examples), such a Lie algebra emerges directly. It is shown below that in this case $\{h_X, h_Y\} := X \lrcorner Y \lrcorner \Omega$ is hamiltonian for $[X, Y]$ and $h_{[X,Y]} = \mathrm{d}(X \lrcorner h_Y) + \{h_X, h_Y\} = \mathrm{d}(X \lrcorner Y \lrcorner \Theta) + \{h_X, h_Y\}$. If the basis vector fields of a Lie algebra of infinitesimal exact multisymplectomorphisms have the property of mutual Θ-orthogonality, $X_a \lrcorner X_b \lrcorner \Theta = 0$ or a constant, then we would have the Lie algebra map property $h_{[X_a,X_b]} = \{h_a, h_b\}$.

Multi-Poisson action

Then, up to a certain cohomological obstruction, (if $h_{[\eta^1_{\mathbb{M}},\eta^2_{\mathbb{M}}]} - \{h_{\eta^1_{\mathbb{M}}}, h_{\eta^1_{\mathbb{M}}}\} = \omega(\eta^1_{\mathbb{M}}, \eta^2_{\mathbb{M}})$, for some closed $d-1$-form ω on $\mathbb{M}$) there exists a hamiltonian map $\tilde{h}$ such that there is a Lie algebra homomorphism $\tilde{h} : \mathfrak{g} \longrightarrow \tilde{h}(\mathfrak{g}) \subset (\Lambda^{d-1}\mathbb{M}, \{\})$ from the Lie algebra $\mathfrak{g}$ of the transformation group to the multi-Poisson algebra of spacetime $d-1$-forms on the multisymplectic manifold $\mathbb{M}$. In the case of infinitesimal exact multisymplectomorphisms the obstruction is $\omega = \mathrm{d}(X_{\eta^1} \lrcorner X_{\eta^2} \lrcorner \Theta)$. Because the hamiltonian form for a given hamiltonian vector field is only defined up to addition with closed form, there may be ways to choose this closed form to remove the obstruction.

Such constraints $\tilde{h}_\eta$ form a Lie algebra $(T, \{\cdot,\cdot\})$. If (η_a) is a basis of $\mathfrak{g}$, then $\tilde{h}_a := \tilde{h}_{\eta_a}$ are a basis of the constraint space $\tilde{h}(\mathfrak{g})$, and $\{\tilde{h}_a, \tilde{h}_b\} = f^c_{ab}\tilde{h}_c$, where f^c_{ab} are the structure constants for the Lie algebra $\mathfrak{g}$ in the basis (η_a). This is called a first class set of constraints following the Dirac terminology, which are constraints whose multiPoisson algebra close on the constraints, which is the case here. Such an action is called a multiPoisson action of a Lie algebra on a multisymplectic manifold.

Moment map

The transpose of the above map, from the Lie algebra to the multi-Poisson algebra of the constraints, is the map from the multisymplectic manifold to the Lie algebra dual, and is called the moment map: $\mathfrak{J} : \mathbb{M} \longrightarrow \mathfrak{g}^* :: m \mapsto v = \mathfrak{J}(m)$, where $\mathfrak{J}$ is defined by $h_\eta(m) = \langle \mathfrak{J}(m), \eta \rangle$. If there is a Lie algebra homomorphism as above then the moment map intertwines between the canonical coadjoint action of G on $\mathfrak{g}^*$ and the Lie group action of G on $\mathbb{M}$ (this is termed G-equivariance).

Symmetry of the DDW Hamiltonian

If the dynamical DDW Hamiltonian function $\mathcal{H}$ of a field theory, which generates the evolution (i.e. $\partial_\mu u^i = -\{\mathcal{H}, u^i\}_\mu$, $\partial_\mu p_i^\mu = -\{\mathcal{H}, p_i^\mu\}_\mu$, are the DDW equations of motion of the field), is invariant ($X_a(\mathcal{H}) = 0$) under the hamiltonian Lie group G, with Lie algebra hamiltonian action $X_a = \rho(\eta_a)$ and constraints h_a, where η_a are a basis of the Lie algebra and ρ is the action (diffeomorphism) on the manifold $\mathbb{M}$, then the DDW Hamiltonian $\mathcal{H}$ is said to be symmetric under the action ρ of the Lie group G. Then, corresponding to the hamiltonian vector fields X_a, there will be a hamiltonian functions h_a with $X_a \lrcorner \Omega = \mathrm{d}h_a$ and,

$$0 = X_a(\mathcal{H}) = X_a \lrcorner \mathrm{d}\mathcal{H} = \{h_a, \mathcal{H}\} = X_a \lrcorner X_{\mathcal{H}} \lrcorner \Omega = (-1)^d X_{\mathcal{H}} \lrcorner \mathrm{d}h_a \tag{3.89}$$

where $X_{\mathcal{H}}$ is a horizontal d multivector field on a submanifold of multiphase space which is the tangent to any solution of the DDW field equations viewed as a section of multiphase space over spacetime (see appendix E 'Multivector Picture' which examines such field solutions which satisfy $X_{\mathcal{H}} \lrcorner \Omega = \mathrm{d}\mathcal{H}$).

Reduction

One may choose a particular value of h: $h_a = 0$, for all a, and consider the locus of points $\mathbb{M}_G \subset \mathbb{M}$ which are solutions of this constraint equation, the zero level set of the h's in $\mathbb{M}$. $\mathbb{M}_G$ is the same as $\mathfrak{J}^{-1}(0)$, the kernel of the moment map. If $\bar{0}$ is a regular value of h, i.e. the Jacobean matrix of h has constant rank K on the zero locus, then $\mathbb{M}_G$ is a coisotropic submanifold of $\mathbb{M}$. (A submanifold $\mathbb{M}_c$ of a multisymplectic manifold $(\mathbb{M}, \Omega)$ is called *coisotropic* if the kernel $\mathfrak{T}\mathbb{M}_c^\perp$ of Ω_c lies inside $\mathfrak{T}\mathbb{M}_c$: $\mathfrak{T}\mathbb{M}_c^\perp \subset \mathfrak{T}\mathbb{M}_c$, where $Y \in \mathfrak{T}\mathbb{M}_c^\perp$ iff $Y \lrcorner \Omega \llcorner X = 0 \;\; \forall X \in \mathfrak{T}\mathbb{M}_c$, and where

Ω_c is the form Ω restricted to $\mathfrak{T}\mathbb{M}_c$). This is because $0 = Y \lrcorner \Omega \llcorner X_h = Y \lrcorner \mathrm{d}h$ implies that Y lies in the $h =$ constant hypersurface, in this case the zero locus. If the dimension of the kernel in a coisotropic submanifold $\mathbb{M}_c$ is constant on $\mathbb{M}_c$, then the tangent spaces $\mathfrak{T}_m\mathbb{M}_c^{\perp} \subset \mathfrak{T}_m\mathbb{M}_c$ ranging over $m \in \mathbb{M}_c$ define a distribution called the characteristic distribution of Ω_c, and this distribution is Frobenius integrable (shown next), and as a consequence the coisotropic submanifold foliates into connected submanifolds whose tangent spaces are this distribution. The space of leaves will be the reduced multisymplectic manifold we are seeking.

The Frobenius condition, $[X_a, X_b] \in \mathfrak{T}\mathbb{M}_c^{\perp}, \forall$ local sections (i.e. vector fields in $\mathbb{M}_c$) $X_a, X_b \in \mathfrak{T}\mathbb{M}_c^{\perp}$, is a consequence of the closure of the premultisymplectic form Ω_c: If X_a, X_b are any local sections in $\mathfrak{T}\mathbb{M}_c^{\perp}$ and Y_i are any local section in $\mathfrak{T}\mathbb{M}_c$ then we can use the identity on $d+1$-forms ω:

$$
\begin{aligned}
(\mathrm{d}\omega)(X_0, \dots, X_{d+1}) \; = \\
= \sum_{i=1}^{d+1} (-1)^i X_i(\omega(X_0, \dots, \hat{X}_i, \dots, X_{d+1})) + \sum_{i,j=1}^{d+1} (-1)^{i+j} \omega([X_i, X_j], \dots, \hat{X}_i, \dots, \hat{X}_j, \dots)
\end{aligned}
\tag{3.90}
$$

to obtain

$$
\begin{aligned}
(\mathrm{d}\Omega_c)(X_a, X_b, Y_0, \dots, Y_{d-1}) \; &= \; \sum_{i=1}^{d-1} (-1)^i Y_i(\Omega_c(X_a, X_b, Y_0, \dots, \hat{Y}_i, \dots, Y_{d-1})) \\
&+ X_a(\Omega_c(X_b, Y_0, \dots, Y_{d-2})) - X_b(\Omega_c(X_a, Y_0, \dots, Y_{d-1})) \\
&+ \sum_{i=1}^{d-1} (-1)^i \Omega_c([X_a, Y_i], X_b, , Y_0, \dots, \hat{Y}_i, \dots, Y_{d-1}) \\
&- \sum_{i=1}^{d-1} (-1)^i \Omega_c([X_b, Y_i], X_a, Y_0, \dots, \hat{Y}_i, \dots, Y_{d-1}) - \Omega_c([X_a, X_b], Y_0, \dots, Y_{d-2})
\end{aligned}
\tag{3.91}
$$

but, because X_a and X_b are in the kernel of Ω_c, only the last term on the right hand side is non-zero and so

$$
(\mathrm{d}\Omega_c)(X_a, X_b, Y_0, \dots, Y_{d-1}) = -\Omega_c([X_a, X_b], Y_0, \dots, Y_{d-2})
\tag{3.92}
$$

but because of closure, $\mathrm{d}\Omega_c = 0$, and so $\Omega_c([X_a, X_b], Y_0, \dots, Y_{d-2}) = 0$ for all local sections Y_i in $\mathfrak{T}\mathbb{M}_c$, therefore $[X_a, X_b] \in \mathfrak{T}\mathbb{M}_c^{\perp}$. Because the X_a's are a basis of the Lie algebra action, $[X_a, X_b] = f_{ab}^k X_k$, so the X_a's are involutive and thus can be integrated to foliate $\mathbb{M}_G$.

If the action of G is free and proper, the space of leaves $\mathbb{M}_{GG} := M//G := \mathbb{M}_G/G$ is a manifold as indicated above, and in fact this manifold has a well defined multisymplectic form $\Omega|_{\mathbb{M}//G}$. (If a multiPoisson action is not free and proper, then it will be locally free and the space of leaves will be an orbifold.) $\Omega|_{\mathbb{M}//G}$ is a multisymplectic form on the reduced space $\mathbb{M}//G$ because the premultisymplectic form $\Omega|_{\mathbb{M}/G}$ on $\mathbb{M}_G$, which is the multisymplectic form Ω in $\mathbb{M}$ pulled back to $\mathbb{M}_g$ by the embedding map (and which preserves the closure property of Ω), has as kernel precisely the characteristic vectors (which are tangent to the leaves of the foliation). So the null directions to the premultisymplectic form are mod-ed out in $\mathbb{M}_G/G$, and the space of leaves thereby becomes a multisymplectic manifold, the multisymplectic quotient of $\mathbb{M}$ by G also known as 'the reduced multiphase space', $(\mathbb{M}//G, \Omega|_{\mathbb{M}//G})$. This process is the generalization to multiphase space of Marsden-Weinstein symplectic reduction [6].

Examples

Example 1: Exact multisymplectomorphisms

We consider an exact multisymplectic manifold (with an exact multisymplectic form $\Omega = -\mathrm{d}\Theta$). Let $h_X = X \lrcorner \Theta$

Here we employ the identities $\mathcal{L}_X \mathrm{d}\omega = \mathrm{d}\mathcal{L}_X\omega$ and $\mathcal{L}_X\omega = \mathrm{d}(X \lrcorner \omega) + X \lrcorner (\mathrm{d}\omega)$ for any form ω and any vector field X: Then, if X is an infinitesimal multisymplectomorphism ($\mathcal{L}_X\Omega = 0$):

$$0 = \mathcal{L}_X\Omega = \mathcal{L}_X(-\mathrm{d}\Theta) = -\mathrm{d}\mathcal{L}_X\Theta = -\mathrm{d}(\mathrm{d}(X \lrcorner \Theta) + X \lrcorner (\mathrm{d}\Theta)) = -\mathrm{d}(\mathrm{d}h_X + X \lrcorner (-\Omega)) \quad (3.93)$$

So $X \lrcorner \Omega$ is closed. If X is an infinitesimal exact multisymplectomorphism, $\mathcal{L}_X\Theta = 0$, then $\mathrm{d}h_X - X \lrcorner \Omega = 0$, $X \lrcorner \Omega$ is exact and X is hamiltonian with hamiltonian $d-1$-form h_X.

The conclusion is that if X is an infinitesimal exact multisymplectomorphism, the hamiltonian $d-1$-form h_X can be simply constructed as $h_X = X \lrcorner \Theta$.

In addition, if X and Y are infinitesimal exact multisymplectomorphisms then $\mathcal{L}_{[X,Y]}\Theta = [\mathcal{L}_X, \mathcal{L}_Y]\Theta = 0$, so $[X, Y]$ is an infinitesimal exact multisymplectomorphism and so $h_{[X,Y]} = [X, Y] \lrcorner \Theta$ is the hamiltonian $d-1$-form for the vector field $[X, Y]$.

Furthermore, X and Y are multi-Poisson, because

$$h_{[X,Y]} = [X, Y] \lrcorner \Theta = [\mathcal{L}_X, i_Y]\Theta = \mathcal{L}_X(i_Y\Theta) - i_Y(\mathcal{L}_X\Theta) = \mathcal{L}_X(h_Y) + 0 =$$

$$= \mathrm{d}(X \lrcorner h_Y) + X \lrcorner (\mathrm{d}h_Y) = \mathrm{d}(X \lrcorner Y \lrcorner \Theta) + X \lrcorner Y \lrcorner \Omega = \mathrm{d}(X \lrcorner h_Y) + \{h_X, h_Y\} \quad (3.94)$$

Taking the exterior derivative on both sides gives: $[X, Y] \lrcorner \Omega = \mathrm{d}h_{[X,Y]} = \mathrm{d}(\mathrm{d}(X \lrcorner h_Y) + \{h_X, h_Y\}) = \mathrm{d}\{h_X, h_Y\}$. Thus $\{h_X, h_Y\} := X \lrcorner Y \lrcorner \Omega$ is hamiltonian for $[X, Y]$ and $h_{[X,Y]} = \mathrm{d}(X \lrcorner h_Y) + \{h_X, h_Y\} = \mathrm{d}(X \lrcorner Y \lrcorner \Theta) + \{h_X, h_Y\}$. If the basis vector fields of a Lie algebra of infinitesimal exact multisymplectomorphisms have the property of mutual Θ-orthogonality, $X_a \lrcorner X_b \lrcorner \Theta = 0$, or a constant, then we would have $h_{[X_a,X_b]} = \{h_a, h_b\}$.

In multiphase space $\Theta = p_i^\alpha \mathrm{d}q^i \wedge \mathrm{d}^{d-1}x_\alpha$.

The gauge variations in electromagnetism and Yang-Mills (below) are infinitesimal exact multisymplectomorphisms.

Example 2: Action on configuration space.

An example of a multiPoisson action is any Lie group diffeomorphic action G acting vertically on the fibers of a configuration bundle $\mathcal{E} \longrightarrow B$ naturally and equivariantly extended to the multiphase space $\pi^{\mathcal{E},J^1\mathcal{E}^*} : J^1\mathcal{E}^* \longrightarrow \mathcal{E}$. The reduced phase space is $J^1\mathcal{E}^*//G = J^1(\mathcal{E}/G)^*$, if the action is free and proper on $\mathcal{E}$.

3.6.4 Multiphase-space action variation expressed using extended multi-Poisson brackets

Repeating (3.55), a particular infinitesimal variation δ_Y of a path, $(u^i(x), p^\mu_i(x))$, in multiphase space results in the following variation in the multiphase-space action:

$$\delta_Y S_{MP} \;=\; \delta_Y \int_{\Gamma J^1\mathcal{E}^*} (p^\mu_i \partial_\mu u^i - \mathcal{H}) \mathrm{d}^d x \;=$$

$$= \int_{\Gamma J^1\mathcal{E}^*} [\,(\partial_\mu u^i - \frac{\partial \mathcal{H}}{\partial p^\mu_i})\delta_Y p^\mu_i - (\partial_\mu p^\mu_i + \frac{\partial \mathcal{H}}{\partial u^i})\delta_Y u^i + \frac{\partial \mathcal{H}}{\partial x^\mu}\delta_Y x^\mu + \partial_\mu(\,\delta_Y u^i p^\mu_i + \mathcal{L}_{MP}\,\delta_Y x^\mu\,)\,]\,\mathrm{d}^d x$$

$$= \int_{\Gamma J^1\mathcal{E}^*} [\; -\widetilde{\{}u^i, H\}_\mu \,\delta_Y p^\mu_i \;+\; \widetilde{\{}p^\mu_i \mathrm{d}x_\mu, H\}\delta_Y u^i \;-\; \widetilde{\{}p\;,\;H\,\}_\mu \delta_Y x^\mu\,]\;\mathrm{d}^d x$$

$$+ \int_{\partial\Gamma J^1\mathcal{E}^*} (\,\delta_Y u^i p^\mu_i \;+\; \mathcal{L}_{MP}\;\delta_Y x^\mu\,)\mathrm{d}S_\mu$$

$$= \int_{\Gamma J^1\mathcal{E}^*} [\; -\widetilde{\{}u^i\;,\;\widetilde{T}^\mu{}_\nu \mathrm{d}x_\mu\,\}\;\delta_Y p^\nu_i \;+\; \widetilde{\{}p^\nu_i \mathrm{d}x_\nu\;,\;\widetilde{T}^\mu{}_\nu \mathrm{d}x_\mu\,\}\delta_Y u^i \;-\; \widetilde{\{}p\;,\;\widetilde{T}^\mu{}_\nu \mathrm{d}x_\mu\,\}\delta_Y x^\nu\,]\;\mathrm{d}^d x$$

$$+ \int_{\partial\Gamma J^1\mathcal{E}^*} (\,\delta_Y u^i p^\mu_i \;+\; \mathcal{L}_{MP}\;\delta_Y x^\mu\,)\mathrm{d}S_\mu \tag{3.95}$$

If we assume the variation of the action is zero for the variation δ_Y of the path $(u^i(x), p^\mu_i(x))$, then

$$\partial_\mu(\,\delta_Y u^i p^\mu_i \;+\; \mathcal{L}_{MP}\;\delta_Y x^\mu\,) \;=\; -(\partial_\mu u^i - \frac{\partial \mathcal{H}}{\partial p^\mu_i})\delta_Y p^\mu_i + (\partial_\mu p^\mu_i + \frac{\partial \mathcal{H}}{\partial u^i})\delta_Y u^i \;-\; \frac{\partial \mathcal{H}}{\partial x^\mu}\delta_Y x^\mu \tag{3.96}$$

On shell, when the path $(u^i(x), p_i^\mu(x))$ satisfies the DDW equations, we have

$$\partial_\mu(\,\delta_Y u^i p_i^\mu \;+\; \mathcal{L}_{MP}\;\delta_Y x^\mu\,) \;\approx\; -\,\frac{\partial \mathcal{H}}{\partial x^\mu}\delta_Y x^\mu \tag{3.97}$$

If, in addition, the DDW Hamiltonian is not explicitly a function of spacetime, $\frac{\partial \mathcal{H}}{\partial x^\mu} = 0$, or the variation is vertical in the multiphase-space fiber and not in spacetime position, $\delta_Y x^\mu = 0$, then

$$\partial_\mu(\,\delta_Y u^i p_i^\mu \;+\; \mathcal{L}_{MP}\;\delta_Y x^\mu\,) \;\approx\; 0 \tag{3.98}$$

and so $J^\mu := \delta_Y u^i p_i^\mu \;+\; \mathcal{L}_{MP}\;\delta_Y x^\mu$ is a conserved current.

If the infinitesimal symmetry is a Lie algebra $\mathfrak{g} = \{\epsilon^a \delta_a\}$ of symmetries where $\{\delta_a\}$ is a basis and ϵ^a are continuous parameters, and we have a global symmetry in that the parameters does not vary over spacetime, then

$$\partial_\mu(\,\epsilon^a\delta_a u^i p_i^\mu \;+\; \mathcal{L}_{MP}\;\epsilon^a\delta_a x^\mu\,) \;\;=\;\; \epsilon^a\;\partial_\mu(\,\delta_a u^i p_i^\mu \;+\; \mathcal{L}_{MP}\;\delta_a x^\mu\,) \;\approx\; 0 \tag{3.99}$$

and $J_a^\mu := \delta_a u^i p_i^\mu \;+\; \mathcal{L}_{MP}\;\delta_a x^\mu$ are conserved currents under the conditions specified in the previous sentence. This implies that, in Minkowski space, $\int_{\text{time slice}} J_a^0\;\mathrm{d}^{d-1}x \approx Q$, a constant of motion, $\dot{Q} \approx 0$, whose value depends on the initial conditions (but may change with a change of frame).

The current J_a^μ is a geometrical object described best as a dual Lie algebra valued spacetime $d-1$-form: $J(x) := J_a^\mu(x)\omega^a \otimes \mathrm{d}^{d-1}x_\mu$.

The map $B \longrightarrow \mathfrak{g}^* \bigotimes \Lambda^{d-1}B :: x \mapsto J_a^\mu(x)\omega^a \otimes \mathrm{d}^{d-1}x_\mu$, is the multimoment map, the generalization of the moment map in symplectic mechanics. ($\omega^a \in \mathfrak{g}^*$ is the basis element in the Lie algebra dual to $\mathfrak{g}$ such that $\omega^a(\delta_b) = \delta_b^a$.)

In the case of a local symmetry, where there is a bundle of parameter spaces over spacetime rather than a single parameter space such as in a global symmetry, and a particular value of the variation parameter is given by a section of the parameter bundle: $(f^a(x))$, a set of functions over spacetime, corresponding to a particular position dependent variation $\delta_f = f^a(x)\delta_a$.

In this case the parameter does not commute with the spacetime derivative ∂_μ and so, instead of (3.99), we get

$$\begin{aligned} &\partial_\mu(\,f^a(x)(\delta_a u^i p_i^\mu \;+\; \mathcal{L}_{MP}\;\delta_a x^\mu\,)\,) \\ =\;\; &(\partial_\mu f^a(x))(\delta_a u^i p_i^\mu \;+\; \mathcal{L}_{MP}\;\delta_a x^\mu\,) \;+\; f^a(x)\partial_\mu(\delta_a u^i p_i^\mu \;+\; \mathcal{L}_{MP}\;\delta_a x^\mu\,) \;\;\approx\;\; 0 \end{aligned} \tag{3.100}$$

So $(\partial_\mu f^a(x))J^\mu_a \;+\; f^a(x)\partial_\mu J^\mu_a \;\approx\; 0$. But, because the $f^a(x)$ can be arbitrarily chosen functions of spacetime, we need $J^\mu_a \approx 0$ or, if $f^a(x)$ or the variation δ_a or the fields are zero on the boundary, then we require that $\partial_\mu J^\mu_a = 0$, i.e. $J^\mu_a \approx L^\mu_a$, spacetime constants. Thus the system is constrained to the surface defined by these constraint equations.

If we view the original action as being invariant under a global symmetry, and we want to impose the symmetry as a local symmetry at each point of spacetime, then by restricting the motion to the constraint surface in multiphase space defined by $J^\mu_a = 0$, we can ensure that the local gauge symmetry holds. Compatibility with a DDW Hamiltonian may force further constraints.

3.7 Summary of examples of multiphase-space systems

This is a list of some of the examples used to illustrate multiphase-space systems, indicating in which sections they are to be found. The Yang-Mills example is in section 3.8, while the other examples are in Appendix D, 'Other multiphase-space examples'.

(A) Current linear in the multimomenta:

---

1) Scalar fields with global symmetry D.1:

Configuration fields: q^j. Multimomenta p^μ_i

DDW Hamiltonian: $\mathcal{H} \;=\; \frac{1}{2m}\bar{p}^2 \;+\; V(\bar{q}^2) \;=\; \frac{1}{2m}p^\mu_i p^\nu_j \; g_{\mu\nu}\delta^{ij} \;+\; V(\bar{q}^2)$

Current: $J^\mu_Y \;=\; Y^{rs}(\; p^\mu_i(M_{rs})_{ij}q^j \;)$ with global parameters Y^{rs}, and M_{rs} anti-symmetric matrices forming the basis of a commutator Lie algebra of symmetries.

The variations are:

$\delta_Y q^i = Y^{rs}\delta_{rs}q^i = Y^{rs}(M_{rs})_{ij}q^j \quad \delta_Y x^\mu = 0$

and

$\delta_Y p^\mu_i(x) = Y^{rs}\delta_{rs}p^\mu_i(x) = Y^{rs}(M_{rs})_{ji}p^\mu_j(x) = Y^{rs}(M^T_{rs})_{ij}p^\mu_j(x) = -Y^{rs}(M_{rs})_{ij}p^\mu_j(x)$

---

2) The Electromagnetic Field D.2, phase-space BRST 4.3.1, multiphase-space BRST 4.7:

Configuration fields: A_ν . Multimomenta $p^{\mu\nu}$

DDW Hamiltonian: $\mathcal{H} = p^{\mu\nu}\partial_\mu A_\nu \; - L = \frac{1}{2}p^{[\mu\nu]}p_{[\mu\nu]} + p^{(\mu\nu)}\partial_\mu A_\nu \; = \frac{1}{2}p^{[\mu\nu]}p_{[\mu\nu]} - \partial_\mu p^{(\mu\nu)}A_\nu$

Current: $J^\mu_f = \delta_f A_\nu p^{(\mu\nu)} = \partial_\nu f p^{(\mu\nu)}$ where $f(x)$ is an arbitrary function on spacetime.

The variations are: $\delta_f A_\rho = -\partial_\rho f(x)$, $\delta_f p^{\mu\beta} = 0$, $\delta_f x^\mu = 0$

---

3) Non Abelian Yang-Mills 3.8 , multiphase-space BRST 4.7.2 :

Configuration fields: A^a_ν. Multimomenta $p^{\mu\nu}_a$

DDW Hamiltonian: $\mathcal{H} = p^{\mu\nu}_a\partial_\mu A^a_\nu \; - \mathcal{L} = \frac{1}{2}p^{[\mu\nu]}_a p^a_{[\mu\nu]} - \frac{g}{2}p^{[\mu\nu]}_a f^a_{bc}A^b_\mu A^c_\nu + p^{(\mu\nu)}_a\partial_\mu A^a_\nu$

Current: $J^\mu_f = \delta_f A^a_\nu p^{(\mu\nu)}_a = (-D_\nu f^a)p^{(\mu\nu)}_a$

The variations are: $\delta_f A^a_\rho = -D_\rho f^a(x)$, $\delta_f p^{\mu\beta}_b = -g f^c f^a_{bc} p^{(\mu\beta)}_a$, $\delta_f x^\mu = 0$

(B) Current quadratic in the multimomenta:

---

4) The bosonic string D.3:

Configuration fields: $X^i, h_{\alpha\beta}$. Multimomenta $p^\gamma_i, H^{\gamma\alpha\beta}$

DDW Hamiltonian: $\mathcal{H} = -[\frac{1}{4}T^{-1}h^{-\frac{1}{2}}h_{\alpha\beta}p^\alpha_i p^\beta_j g^{ij} + Th^{\frac{1}{2}}(d-2) - H^{\gamma\alpha\beta}\partial_\gamma h_{\alpha\beta}]$

Current: $J^\alpha_f \; = \frac{1}{4}T^{-1}h^{-\frac{1}{2}} \; (\; p^\alpha_i p^\beta_j - \frac{1}{2}h^{\alpha\beta}h_{\alpha'\beta'}p^{\alpha'}_i p^{\beta'}_j)g^{ij}(X)f_\beta(\sigma)$

The variations are $\delta_f h_{\alpha\beta} = \partial_{(\alpha}f_{\beta)}$ where $f_\beta(\sigma)\mathrm{d}x^\beta$ is an arbitrary 1-form, representing an infinitesimal diffeomorphism on the brane surface.

---

5) General Relativity D.4:

Configuration fields: $\bar{g}^{\mu\nu}$. Multimomenta $p^{\gamma}_{\mu\nu}$

DDW Hamiltonian: $\mathcal{H}(\bar{g}^{\mu\nu}, p^{\alpha}_{\mu\beta}) = \bar{g}^{\mu\nu} p^{\alpha}_{\mu\beta} p^{\beta}_{\alpha\nu} + p^{\beta}_{\beta(\nu}\partial_{\mu)}\bar{g}^{\mu\nu} =: \bar{g}^{\mu\nu} p^{2}_{\mu\nu} + p^{\alpha}_{\alpha(\nu}\partial_{\beta)}\bar{g}^{\beta\nu} = \bar{g}^{\mu\nu} p^{2}_{\mu\nu} - \partial_{(\beta} p^{\alpha}_{|\alpha|\nu)}\bar{g}^{\beta\nu}$

Current: $T^{\gamma}_{f} \;=\; -(\; 2\; \bar{g}^{\kappa(\mu} p^{\gamma)}_{\kappa(\nu} p^{\beta}_{|\beta|\mu)} \;+\; \bar{g}^{\gamma\mu} p^{\beta}_{\beta(\nu}\partial_{\mu)}\;)\; f^{\nu}(x)$

The variations are $\delta_f \bar{g}^{\kappa\lambda} = 2\partial_\mu f^{(\lambda}\bar{g}^{\kappa)\mu} + f^{\nu}\partial_\nu \bar{g}^{\lambda\kappa} - \partial_\nu f^{\nu}\bar{g}^{\lambda\kappa} = 2\partial^{(\lambda} f^{\kappa)} + f^{\nu}\partial_\nu \bar{g}^{\lambda\kappa} - \partial_\nu f^{\nu}\bar{g}^{\lambda\kappa}$ and

$\delta_f p^{\lambda}_{\rho\sigma} = -\delta^{\lambda}_{\rho} f^{\nu}\partial_{(\sigma} p^{\beta}_{|\beta|\nu)}$ where $f^{\mu}(x)$ is a vector field representing an infinitesimal diffeomorphism on spacetime.

3.8 Example of multisymplectic field theory: Yang-Mills

Yang-Mills is a model example for the application of multiphase-space methods because of its physical application as well as the fact that it has properties which make multiphase-space methods applicable. Several other multiphase-space examples are explained in Appendix D : 'Other multiphase-space examples'.

The Yang-Mills example here in further developed as an example of multiphase-space BRST in the section 4.7.2 in the chapter on BRST. The special case of electromagnetism is described in appendix D.2 and is further developed as an example of conventional phase-space BRST in section 4.3.1 and as an example of multiphase-space BRST in section 4.7.

3.8.1 Lagrangian analysis

The configuration space action is

$$S[A^{a}_{\mu}(x)] = \int \mathcal{L}\mathrm{d}^{d}x = \int \frac{1}{4} F^{a}_{\mu\nu} F^{b}_{\lambda\rho} g^{\mu\lambda} g^{\nu\rho} t_{ab}\mathrm{d}^{d}x$$
$$= \int |DA|^{2}\mathrm{d}^{d}x = \int D_{[\mu}A^{a}_{\nu]}\; D_{[\lambda}A^{b}_{\rho]}\; g^{\mu\lambda} g^{\nu\rho} t_{ab}\; \mathrm{d}^{d}x \tag{3.101}$$

where $A = (A^a_\mu T_a)\mathrm{d}x^\mu$ is the connection on a vector bundle over Minkowski spacetime of a gauge group G with generators T_a and structure constants f^a_{bc}: $[T_a, T_b] = f^c_{ab}T_c$, and Killing form t_{ab}, which we can assume to be $t_{ab} = \delta_{ab}$. $D = \mathrm{d} - \frac{g}{2}[A_\mu, \cdot] = \mathrm{d} - gf^a_{bc}A^b_\mu T_a \otimes T^{*c}$ is the covariant exterior derivative which acts on Lie algebra valued 1-forms such as $A = (A^a_\mu T_a)\mathrm{d}x^\mu$ above. The covariant derivative is $D_\mu = \mathbf{1}\partial_\mu - gA_\mu = \mathbf{1}\partial_\mu - gA^a_\mu T_a$, where T_a's are basis elements of the Lie algebra which are matrices in the representation of the objects that the derivative is acting on. The factors in the kinetic terms in the Lagrangian density are

$$F^a_{\mu\nu} := 2D_{[\mu}A^a_{\nu]} = 2\partial_{[\mu}A^a_{\nu]} - [A_\mu, A_\nu]^a = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu - gf^a_{bc}A^b_\mu A^c_\nu = [D_\mu, D_\mu]^a \tag{3.102}$$

The action is invariant under variations

$$\delta_f A^a_\mu(x) = -D_\mu f^a(x) = -\partial_\mu f^a + [A_\mu, f]^a = -\partial_\mu f^a - gf^a_{bc}A^b_\mu f^c \tag{3.103}$$

for arbitrary sections f^a of the vector bundle. These are the gauge transformations and the infinitesimal gauge transformation algebra closes even off shell: $[\delta_{f_1}, \delta_{f_2}] = \delta_{f_3}$ where $f^a_3 = f^a_{bc}f^b_1 f^c_2$. The corresponding variation of $F^a_{\mu\nu}$ is $\delta_f F^a_{\mu\nu} = g[F^b_{\mu\nu}, f^c]^a = -gf^a_{bc}F^b_{\mu\nu}f^c$ and $\delta_f F^{\mu\nu}_a = g[F^{\mu\nu}_b, f^c]_a = gf^b_{ac}F^{\mu\nu}_b f^c$, so $F_{\mu\nu}$ transforms in the adjoint representation.

3.8.2 Legendre transformation and Hamiltonian analysis

The Legendre transformation maps to multimomenta:

$$p^{\mu\nu}_a \approx\ 2D_{[\lambda}A^b_{\rho]}\ g^{\mu\lambda}g^{\nu\rho}\,\eta_{ba}\ =:\ F^{\mu\nu}_a \tag{3.104}$$

and primary constraints:

$$p^{(\mu\nu)}_a \approx\ 0 \tag{3.105}$$

The DDW Hamiltonian is

$$\begin{aligned}\mathcal{H} = p^{\mu\nu}_a\partial_\mu A^a_\nu\ - \mathcal{L} &= \frac{1}{4}p^{[\mu\nu]}_a p^a_{[\mu\nu]} - \frac{g}{2}p^{[\mu\nu]}_a f^a_{bc}A^b_\mu A^c_\nu + p^{(\mu\nu)}_a\partial_\mu A^a_\nu \\ &= \frac{1}{4}p^{[\mu\nu]}_a p^a_{[\mu\nu]} - \frac{g}{2}p^{[\mu\nu]}_a f^a_{bc}A^b_\mu A^c_\nu - \partial_\mu p^{(\mu\nu)}_a A^a_\nu \\ &= \frac{1}{4}p^{[\mu\nu]}_a p^a_{[\mu\nu]} - \frac{g}{2}p^{[\mu\nu]}_a f^a_{bc}A^b_\mu A^c_\nu - D_\mu p^{(\mu\nu)}_a A^a_\nu\end{aligned} \tag{3.106}$$

employing integration by parts inside the first order action (4.122) for the last equality and using $\partial_\mu p^{(\mu\nu)}_a A^a_\nu = D_\mu p^{(\mu\nu)}_a A^a_\nu$. The presence of a term in the Hamiltonian incorporating derivatives, $\partial_\mu A^a_\nu$, of the fields is usual when there is a primary constraint and the Legendre transformation is not fully invertible, which results in it not being possible to replace all the derivatives of the fields with multimomenta.

On the primary constraint surface $T_a^{(1)(\mu\nu)} = p_a^{(\mu\nu)} \approx 0$, the DDW Hamiltonian is

$$\mathcal{H} \stackrel{c}{=} \frac{1}{4} p_a^{[\mu\nu]} p^a_{[\mu\nu]} - \frac{g}{2} p_a^{[\mu\nu]} [A_\mu, A_\nu]^a = \frac{1}{4} p_a^{[\mu\nu]} p_b^{[\lambda\rho]} g_{\mu\lambda} g_{\nu\rho} t^{ab} - \frac{g}{2} p_a^{[\mu\nu]} f^a_{bc} A^b_\mu A^c_\nu \tag{3.107}$$

The gauge variation of the DDW Hamiltonian is

$$\begin{aligned} &\delta_f \mathcal{H} = \\ &\delta_f(\frac{1}{4} p_a^{[\mu\nu]} p^a_{[\mu\nu]}) - \frac{g}{2} \delta_f p_a^{[\mu\nu]} f^a_{bc} A^b_\mu A^c_\nu - g p_a^{[\mu\nu]} f^a_{bc} \delta_f A^b_\mu A^c_\nu - \partial_\mu \delta_f p_a^{(\mu\nu)} A^a_\nu - \partial_\mu p_a^{(\mu\nu)} \delta_f A^a_\nu \\ &= \partial_\mu f^b (g f^a_{bc} A^c_\nu p_a^{[\mu\nu]} + \partial_\nu p_b^{(\nu\mu)}) = \partial_\mu f^b (g f^a_{bc} A^c_\nu p_a^{[\mu\nu]} + D_\nu p_b^{(\nu\mu)}) \stackrel{c}{=} g f^a_{bc} A^c_\nu \partial_\mu f^b p_a^{\mu\nu} \end{aligned} \tag{3.108}$$

The last equality '$\stackrel{c}{=}$' holds on the primary constraint surface. If the DDW Hamiltonian is to be gauge invariant we need this to be zero for all values of the gauge parameters $f(x)$, so there is a secondary constraint $T_b^2 = (g p_a^{[\alpha\mu]} f^a_{bc} A^c_\alpha) \mathrm{d}^{d-1} x_\mu$ where $\partial_\mu (g p_a^{[\mu\alpha]} f^a_{bc} A^c_\alpha) = 0$, which can be written $\partial_\mu \langle [\cdot, A_\alpha], p^{[\mu\alpha]} \rangle = 0$.

Calculating $\delta_f \mathcal{H}$ assuming the invariance of the Lagrangian density,

$$\begin{aligned} \delta_f \mathcal{H} = \delta_f (p_a^{\mu\nu} \partial_\mu A^a_\nu - \mathcal{L}) &= \delta_f p_a^{\mu\nu} \partial_\mu A^a_\nu + p_a^{\mu\nu} \partial_\mu \delta_f A^a_\nu - \delta_f \mathcal{L} \\ &= -p_a^{\mu\nu} \partial_\mu \partial_\nu f^a + g f^a_{bc} A^c_\nu \partial_\mu f^b p_a^{\mu\nu} - 0 \\ &= -\partial_\mu \partial_\nu f^a p_a^{(\mu\nu)} - \partial_\mu f^b g p_a^{\mu\nu} f^a_{cb} A^c_\nu = -p_a^{\mu\nu} D_\mu (\partial_\nu f^a) \end{aligned} \tag{3.109}$$

assuming $p_a^{(\mu\nu)} = 0$, resulting in the same secondary constraint.

The multi-bracket between the constraints give the desired results (3.84)-(3.86):

$$\begin{aligned} \{T_a^{1\alpha}, T_b^{1\beta}\} &= \{p_a^{(\mu\alpha)} \mathrm{d}^{d-1} x_\mu \ , \ p_b^{(\mu'\beta)} \mathrm{d}^{d-1} x_{\mu'}\} = 0 \\ \{T_a^{1\alpha}, T_b^2\} &= \{p_a^{(\mu\alpha)} \mathrm{d}^{d-1} x_\mu \ , \ g p_{a'}^{\mu\alpha'} f^{a'}_{bc} A^c_{\alpha'} \mathrm{d}^{d-1} x_\mu\} \\ &= - g f^c_{ab} p_c^{(\mu\alpha)} \mathrm{d}^{d-1} x_\mu = -g f^c_{ab} T_c^{1\alpha} \\ \{T_a^2, T_b^2\} &= \{g p_{a'}^{\mu\alpha} f^{a'}_{ac} A^c_\alpha \mathrm{d}^{d-1} x_\mu \ , \ g p_{a''}^{\mu'\alpha'} f^{a''}_{bc'} A^{c'}_{\alpha'} \mathrm{d}^{d-1} x_{\mu'}\} \\ = g^2 p_d^{\mu\alpha} (f^d_{ac} f^c_{be} - f^c_{ae} f^d_{bc}) A^e_\alpha \mathrm{d}^{d-1} x_\mu &= g^2 p_d^{\mu\alpha} f^c_{ab} f^d_{ce} A^e_\alpha \mathrm{d}^{d-1} x_\mu = g f^c_{ab} T_c^2 \end{aligned} \tag{3.110}$$

employing the identity $f^c_{be} f^d_{ac} + f^c_{ea} f^d_{bc} + f^c_{ab} f^d_{ec} = 0$ for structure constants in the last line.

Note that

$$\begin{aligned} \{p_a^{(\mu\alpha)} \mathrm{d}^{d-1} x_\mu \ , \ g p_{a'}^{[\mu\alpha']} f^{a'}_{bc} A^c_{\alpha'} \mathrm{d}^{d-1} x_\mu\} &= 0 \\ \{p_a^{(\mu\alpha)} \mathrm{d}^{d-1} x_\mu \ , \ g p_{a'}^{(\mu\alpha')} f^{a'}_{bc} A^c_{\alpha'} \mathrm{d}^{d-1} x_\mu\} &= - g p_c^{(\mu\alpha)} f^c_{ab} \mathrm{d}^{d-1} x_\mu = -g f^c_{ab} T_c^{1\alpha} \\ \{p_a^{[\mu\alpha]} \mathrm{d}^{d-1} x_\mu \ , \ g p_{a'}^{[\mu\alpha']} f^{a'}_{bc} A^c_{\alpha'} \mathrm{d}^{d-1} x_\mu\} &= - g p_c^{[\mu\alpha]} f^c_{ab} \mathrm{d}^{d-1} x_\mu \\ \{p_a^{\mu\alpha} \mathrm{d}^{d-1} x_\mu \ , T_b^2\} = \{p_a^{\mu\alpha} \mathrm{d}^{d-1} x_\mu \ , \ g p_{a'}^{\mu\alpha'} f^{a'}_{bc} A^c_{\alpha'} \mathrm{d}^{d-1} x_\mu\} &= -g p_c^{\mu\alpha} f^c_{ab} \mathrm{d}^{d-1} x_\mu \end{aligned}$$

$$
\begin{aligned}
\{A^a_\alpha, T^{1\beta}_b\} = \{A^a_\alpha,\ p^{(\mu'\beta)}_b\ \mathrm{d}^{d-1}x_{\mu'}\} \ &= \ \delta^a_b\delta^\beta_\alpha\frac{1}{2}(d+1) \\
\{A^a_\alpha, T^2_b\} = \{A^a_\alpha\ ,\ gp^{\mu\alpha'}_{a'} f^{a'}_{bc} A^c_{\alpha'}\ \mathrm{d}^{d-1}x_\mu\} \ &= \ gf^a_{bc}A^c_\alpha \\
\delta_f A^a_\alpha = \{A^a_\alpha, (\partial_\beta f^b)T^{1\beta}_b - f^b T^2_b\} = \{A^a_\alpha\ ,\ gp^{\mu\alpha'}_{a'} f^{a'}_{bc} A^c_{\alpha'}\ \mathrm{d}^{d-1}x_\mu\} & \\
= \ \partial_\alpha f^a - gf^b f^a_{bc} A^c_\alpha \ &= \ D_\alpha f^a \\
\delta_f \bar{p}^\alpha_a = \{p^{\mu\alpha}_a\ \mathrm{d}^{d-1}x_\mu\ , (\partial_\beta f^b)T^{1\beta}_b - f^b T^2_b\} = \{p^{\mu\alpha}_a\ \mathrm{d}^{d-1}x_\mu\ ,\ -f^b gp^{\mu\alpha'}_{a'} f^{a'}_{bc} A^c_{\alpha'}\ \mathrm{d}^{d-1}x_\mu\} & \\
= \ gf^b p^{\mu\alpha}_c f^c_{ab}\ \mathrm{d}^{d-1}x_\mu &
\end{aligned} \tag{3.111}
$$

The DeDonder Weyl equations of motion are:

$$
\begin{aligned}
\partial_\mu A^a_\nu(x) - \frac{\partial\mathcal{H}}{\partial p^{\mu\nu}_a}(x, A^b_\kappa(x), p^{\lambda\kappa}_b(x)) \ &\approx \ 0 \\
\partial_\mu p^{\mu\nu}_a(x) + \frac{\partial\mathcal{H}}{\partial A^a_\nu}(x, A^b_\kappa(x), p^{\lambda\kappa}_b(x)) \ &\approx \ 0
\end{aligned} \tag{3.112}
$$

which are, substituting for $\mathcal{H}$,

$$
\begin{aligned}
\partial_\mu A^a_\nu - \frac{1}{2}p^a_{[\mu\nu]} + \frac{1}{2}gf^a_{bc}A^b_\mu A^c_\nu - \partial_{(\mu}A^a_{\nu)} \ &\approx \ 0 \\
\partial_\mu p^{\mu\nu}_a - \partial_\mu p^{(\mu\nu)}_a - gp^{[\mu\nu]}_d f^d_{ba} A^b_\mu \ &\approx \ 0
\end{aligned} \tag{3.113}
$$

finally expressed with the covariant derivative, the DDW equations of motion are

$$
\begin{aligned}
D_{[\mu}A^a_{\nu]}(x) - \frac{1}{2}p^a_{[\mu\nu]} \ &\approx \ 0 \\
D_\mu p^{[\mu\nu]}_a \ &\approx \ 0
\end{aligned} \tag{3.114}
$$

which are also the Euler-Lagrange equations for a stationary point of the following first order action:

$$
\begin{aligned}
S_{MP} &= \int_{\Gamma J^1\mathcal{E}^*} (p^{\mu\nu}_a \partial_\mu A^a_\nu - \mathcal{H})\mathrm{d}^d x \\
&= \ \int_{\Gamma J^1\mathcal{E}^*} (p^{\mu\nu}_a \partial_\mu A^a_\nu - \frac{1}{4}p^{[\mu\nu]}_a p^a_{[\mu\nu]} + \frac{1}{2}gf^a_{bc}A^b_\mu A^c_\nu - p^{(\mu\nu)}_a \partial_\mu A^a_\nu\ )\mathrm{d}^d x \\
&= \int_{\Gamma J^1\mathcal{E}^*} (p^{[\mu\nu]}_a D_\mu A^a_\nu - \frac{1}{4}p^{[\mu\nu]}_a p^a_{[\mu\nu]})\mathrm{d}^d x
\end{aligned} \tag{3.115}
$$

which has variation

$$
\begin{aligned}
\delta S_{MP} &= \delta\int_{\Gamma J^1\mathcal{E}^*} (p^{\mu\nu}_a \partial_\mu A^a_\nu - \mathcal{H})\mathrm{d}^d x \\
&= \int_{\Gamma J^1\mathcal{E}^*} (\partial_\mu A^a_\nu - \frac{\partial\mathcal{H}}{\partial p^{\mu\nu}_a})\delta p^{\mu\nu}_a - (\partial_\mu p^{\mu\nu}_a + \frac{\partial\mathcal{H}}{\partial A^a_\nu})\delta A^a_\nu\ \mathrm{d}^d x \ + \ \int_{\partial\Gamma J^1\mathcal{E}^*} [\,\delta A^a_\nu p^{\mu\nu}_a\,]\mathrm{d}S_\mu \\
&= \ \int_{\Gamma J^1\mathcal{E}^*} (\partial_\mu A^a_\nu - \frac{1}{2}p^a_{[\mu\nu]} + \frac{1}{2}gf^a_{bc}A^b_\mu A^c_\nu - \partial_{(\mu}A^a_{\nu)})\delta p^{\mu\nu}_a \\
&\quad -(\partial_\mu p^{\mu\nu}_a - \partial_\mu p^{(\mu\nu)}_a - gp^{[\mu\nu]}_d f^d_{ba} A^b_\mu)\delta A^a_\nu\ \mathrm{d}^d x \ + \ \int_{\partial\Gamma J^1\mathcal{E}^*} [\,\delta A^a_\nu p^{\mu\nu}_a\,]\mathrm{d}S_\mu
\end{aligned} \tag{3.116}
$$

The current in the last term for a gauge transformation is

$$
J^\mu = \delta A^a_\nu p^{\mu\nu}_a \ = -D_\nu f^a p^{\mu\nu}_a \ = -(\partial_\nu f^a + gf^a_{bc}A^b_\nu f^c)p^{\mu\nu}_a \ = -(\partial_\nu f^a T^{(1)\mu\nu}_a + f^c T^{(2)\mu}_c) \tag{3.117}
$$

where $T_a^{(1)\mu} = T_a^{(1)(\mu\nu)}\mathrm{d}x_\nu := p_a^{(\mu\nu)}\mathrm{d}x_\nu$ is the primary constraint and $T_b^2 = T_b^{(2)\mu}\mathrm{d}^{d-1}x_\mu := (gp_a^{[\alpha\mu]}f_{bc}^a A_\alpha^c)\mathrm{d}^{d-1}x_\mu$ is the secondary constraint.

This is a conserved current for a global (spacetime constant) gauge transformation.

By taking the covariant derivative D_λ of the first line in (4.125) above and antisymmetrizing, because $D_{[\lambda}D_\mu A_{\nu]}^a = 0$, we can eliminate the explicit A fields and obtain equations of motion purely in terms of spacetime covariant derivatives (which still depend on A) of the antisymmetric part of the multimomenta:

$$\begin{aligned} D_{[\lambda}p^a_{[\mu\nu]]} &= 0 \\ D_\mu p_a^{[\mu\nu]} &= 0 \end{aligned} \tag{3.118}$$

which are the generalization to Maxwell's equations, where $p^a_{[\mu\nu]} = F^a_{\mu\nu}$, $p^a_{[0i]} = F^a_{0i} = E^a_i$ and $\epsilon_{ijk}p^a_{[jk]} = \epsilon_{ijk}F^a_{jk} = B^a_i$ in the conventional notation for the gluon field.

The infinitesimal gauge variation parameter is the set of functions $f^a(x)$ and the infinitesimal gauge variation is:

$$\delta_f A_\mu^a(x) = -D_\mu f^a(x) = -\partial_\mu f^a + ig[A_\mu, f]^a = -\partial_\mu f^a - g f_{bc}^a A_\mu^b f^c \tag{3.119}$$

and

$$\delta_f p_a^{\mu\nu} = g f_{ac}^b p_b^{\mu\nu} f^c \tag{3.120}$$

This variation of the multimomenta is chosen to be consistent with $p_a^{[\mu\nu]} \approx F_a^{\mu\nu}$.

3.8.3 Constraints as generators of gauge variations

The constraints $D_\mu p_a^{(\mu\nu)} = 0$ generate the gauge transformations $\delta_f A_\rho^b = -D_\rho f^a(x)$, $\delta_f p_b^{\mu\beta} = -gf^c f_{bc}^a p_a^{(\mu\beta)}$, under which the original Lagrangian (4.119) and the first order Lagrangian (4.122) are invariant, via the multi-bracket. We show this by explicitly calculating the variation using the generator:-

Variation of field configuration A:

$$\begin{aligned} -\delta_f A_\rho^b = \{f^a D_\mu p_a^{(\mu\alpha)}, A_\rho^b\}_\alpha \;&=\; -\mathrm{d}_V(f^a D_\mu p_a^{(\mu\alpha)}) \llcorner \Pi_\alpha \lrcorner \mathrm{d}_V(A_\rho^b) \\ &=\; f^a(\partial_\mu p_a^{(\mu\alpha)} - g f_{ba}^c A_\mu^b p_c^{(\mu\alpha)}) \cdot (\frac{\overleftarrow{\partial}}{\partial A_\kappa^c} \wedge \frac{\overrightarrow{\partial}}{\partial p_c^{\kappa\alpha}}) \cdot A_\rho^b \\ &=\; (-\partial_\mu f^a p_a^{(\mu\alpha)} - g f^a f_{ba}^c A_\mu^b p_c^{(\mu\alpha)}) \cdot (\frac{\overleftarrow{\partial}}{\partial A_\kappa^c} \wedge \frac{\overrightarrow{\partial}}{\partial p_c^{\kappa\alpha}}) \cdot A_\rho^b \\ &=\; -(-\partial_\mu f^a - f^c g f_{bc}^a A_\mu^b) \left(\frac{\partial p_a^{(\mu\alpha)}}{\partial p_c^{\kappa\alpha}}\right) \left(\frac{\partial A_\rho^b}{\partial A_\kappa^c}\right) \end{aligned}$$

$$= D_\mu f^a \frac{1}{2}(\delta^\mu_\kappa \delta^\alpha_\alpha + \delta^\alpha_\kappa\ \delta^\mu_\alpha)\delta^c_a \delta^b_c \delta^\kappa_\rho = D_\rho f^b \frac{1}{2}(d+1) \quad (3.121)$$

Variation of multimomenta p:

$$-\delta_f p^{\mu\beta}_b = \{f^a D_\mu p^{(\mu\alpha)}_a, p^{\mu\beta}_b\}_\alpha = f^a(\partial_\mu p^{(\mu\alpha)}_a - g f^c_{ba} A^b_\mu p^{(\mu\alpha)}_c)\cdot(\frac{\overleftarrow{\partial}}{\partial A^c_\kappa} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\kappa\alpha}_c})\cdot p^{\mu\beta}_b$$

$$= (-f^a g f^c_{b'a} p^{(\mu'\alpha)}_c)\ \frac{\partial A^{b'}_{\mu'}}{\partial A^{c'}_\kappa}\frac{\partial p^{\mu\beta}_b}{\partial p^{\kappa\alpha}_{c'}} = (-f^a g f^c_{b'a} p^{(\mu'\alpha)}_c)(\delta^{\mu'}_\kappa\ \delta^{b'}_{c'} \delta^\kappa_\mu \delta^\beta_\alpha \delta^{c'}_b) = -g f^a f^c_{ba} p^{(\mu\beta)}_c \quad (3.122)$$

So $\delta_f p^{(\mu\beta)}_b = g f^c f^a_{bc} p^{(\mu\beta)}_a$, which is zero on the constraint surface $p^{(\mu\nu)}_a = 0$

Variation of the DDW Hamiltonian $\mathcal{H}$:

$$\{\mathcal{H}, f^a D_\mu p^{(\mu\alpha)}_a\}_\alpha$$

$$= \frac{1}{4}p^{[\mu\nu]}_a p^a_{[\mu\nu]} - \frac{g}{2}p^{[\mu\nu]}_a f^a_{bc} A^b_\mu A^c_\nu - \partial_\mu p^{(\mu\nu)}_a A^a_\nu \cdot (\frac{\overleftarrow{\partial}}{\partial A^c_\kappa} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\kappa\alpha}_c}) \cdot f^a(\partial_\mu p^{(\mu\alpha)}_a - g f^c_{ba} A^b_\mu p^{(\mu\alpha)}_c)$$

$$= -(\partial_\lambda p^{(\lambda\kappa)}_a + g p^{[\lambda\kappa]}_c f^c_{ba} A^b_\lambda)\frac{1}{2}(\delta^\mu_\kappa \delta^\alpha_\alpha + \delta^\alpha_\kappa \delta^\mu_\alpha)(-\partial_\mu f^a - g f^c f^a_{bc} A^b_\mu)$$

$$+ [\frac{1}{2}p^a_{[\mu\nu]} - \frac{g}{2} f^a_{bc} A^b_\mu A^c_\nu + \partial_{(\mu} A^a_{\nu)}]\ (-\frac{1}{2})\delta^c_a(\delta^\mu_\kappa \delta^\nu_\alpha - \delta^\nu_\kappa \delta^\mu_\alpha)\delta^\kappa_\mu \delta^b_c[-g f^a f^c_{ba} p^{(\mu\alpha)}_c]$$

$$= -(\partial_\lambda p^{(\lambda\kappa)}_a + g p^{[\lambda\kappa]}_c f^c_{ba} A^b_\lambda)\frac{1}{2}(\delta^\mu_\kappa \delta^\alpha_\alpha + \delta^\alpha_\kappa \delta^\mu_\alpha)(-\partial_\mu f^a - g f^c f^a_{bc} A^b_\mu)$$

$$+ [\frac{1}{2}p^b_{[\kappa\alpha]} - \frac{g}{2} f^b_{ac} A^a_{[\kappa} A^c_{\alpha]} + \partial_{(\kappa} A^b_{\alpha)}]\ (-1)[-g f^a f^c_{ba} p^{(\kappa\alpha)}_c]$$

$$= -D_\lambda p^{(\lambda\mu)}_a \frac{1}{2}(d+1)(-D_\mu f^a) + (\partial_{[\kappa} A^b_{\alpha]} + \partial_{(\kappa} A^b_{\alpha)})\delta_f p^{(\kappa\alpha)}_b$$

$$= -D_\lambda p^{(\lambda\mu)}_a \frac{1}{2}(d+1)\delta_f A^a_\mu + \partial_{(\kappa} A^b_{\alpha)} \delta_f p^{(\kappa\alpha)}_b \quad (3.123)$$

which is zero on the constraint surface $p^{(\mu\nu)}_a = 0$.

Propagation of the constraint $T^{(1)(\mu\alpha)}_a = p^{(\mu\alpha)}_a$:

$$\{\mathcal{H}, T^{(1)(\mu\alpha)}_a\}_\alpha = \{\mathcal{H}, p^{(\mu\alpha)}_a\}_\alpha$$

$$= \frac{1}{4}p^{[\mu\nu]}_a p^a_{[\mu\nu]} - \frac{g}{2}p^{[\mu\nu]}_a f^a_{bc} A^b_\mu A^c_\nu - \partial_\mu p^{(\mu\nu)}_a A^a_\nu \cdot (\frac{\overleftarrow{\partial}}{\partial A^c_\kappa} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\kappa\alpha}_c}) \cdot p^{(\mu\alpha)}_a$$

$$= -(\partial_\lambda p^{(\lambda\kappa)}_a + g p^{[\lambda\kappa]}_c f^c_{ba} A^b_\lambda)\frac{1}{2}(\delta^\mu_\kappa \delta^\alpha_\alpha + \delta^\alpha_\kappa \delta^\mu_\alpha) = -D_\lambda p^{(\lambda\mu)}_a \frac{1}{2}(d+1) \stackrel{c}{=} 0 \quad (3.124)$$

zero on the primary constraint surface (indicated by the equality symbol $\stackrel{c}{=}$). So the gauge variation is consistent with the DDW Hamiltonian.

Repeating the above calculation but with the term $-\partial_\mu p^{(\mu\nu)}_a$ removed from the hamiltonian: $\mathcal{H}_0 = \mathcal{H} - (-\partial_\mu p^{(\mu\nu)}_a) = \frac{1}{4}p^{[\mu\nu]}_a p^a_{[\mu\nu]} - \frac{g}{2}p^{[\mu\nu]}_a f^a_{bc} A^b_\mu A^c_\nu$ gives:

$$\{\mathcal{H}_0, T^{(1)(\mu\alpha)}_a\}_\alpha = \{\mathcal{H}_0, p^{(\mu\alpha)}_a\}_\alpha =$$

$$-(gp_c^{[\lambda\kappa]}f^c_{ba}A^b_\lambda)\frac{1}{2}(\delta^\mu_\kappa\delta^\alpha_\alpha+\delta^\alpha_\kappa\delta^\mu_\alpha)=-(gp_c^{[\lambda\kappa]}f^c_{ba}A^b_\lambda)\frac{1}{2}(d+1)=-T^{(2)\mu}_a\frac{1}{2}(d+1)\ \overset{c}{=}\ 0 \quad (3.125)$$

The infinitesimal gauge variation is

$$\delta_f=-D_\alpha f^a(x)\frac{\partial}{\partial A^a_\alpha}+gf^a(x)f^c_{ba}p^{\alpha\mu}_a\frac{\partial}{\partial p^{\alpha\mu}_b} \quad (3.126)$$

The current produced by the gauge variation is

$$-J_f=(D_\alpha f^a)p^{\alpha\mu}_a\mathrm{d}x_\mu \quad (3.127)$$

This $d-1$ -form current generates the gauge variation on multiphase space via the multibracket:

$$\begin{aligned}X_{J_f}&=\{\cdot,(-D_\alpha f^a)p^{\alpha\mu}_a\mathrm{d}x_\mu\}=\{\cdot,-D_\alpha f^a p^{(\alpha\mu)}_a\mathrm{d}x_\mu\}\\&=(-D_\alpha f^a)\frac{\partial}{\partial A^a_\alpha}+gf^c(x)f^a_{bc}p^{\alpha\mu}_a\frac{\partial}{\partial p^{\alpha\mu}_b}\end{aligned} \quad (3.128)$$

so it has one of the attributes of a hamiltonian $d-1$ -form corresponding to the gauge variation.

The multibracket of the currents for two variations reproduces the Lie algebra:

$$\{J_g,J_f\}=\{(-D_\alpha g^a)p^{\alpha\mu}_a\mathrm{d}x_\mu,(-D_\alpha f^a)p^{\alpha\mu}_a\mathrm{d}x_\mu\}=\mathrm{d}x_\mu(-D_\alpha i[g,f]^a)p^{\mu\alpha}_a=J_{i[g,f]} \quad (3.129)$$

We will now check the hamiltonian property, $\mathrm{d}J_f=X_{J_f}\lrcorner\,\Omega$:

First the left hand side, the exterior derivative of the current, $\mathrm{d}J_f$:

$$\begin{aligned}-\mathrm{d}J_f&=(D_\alpha f^a)\mathrm{d}p^{\alpha\mu}_a\wedge\mathrm{d}x_\mu+(\partial_\beta D_\alpha f^a)p^{\alpha\mu}_a\mathrm{d}x^\beta\wedge\mathrm{d}x_\mu+gf^c_{ba}f^a p^{\alpha\mu}_c\mathrm{d}A^b_\alpha\wedge\mathrm{d}x_\mu\\&=(D_\alpha f^a)\mathrm{d}p^{\alpha\mu}_a\wedge\mathrm{d}x_\mu+(\partial_\beta\partial_\alpha f^a)p^{(\alpha\beta)}_a\mathrm{d}^dx\\&+(gf^c_{ba}A^b_\alpha\partial_\beta f^a)p^{[\alpha\beta]}_c\mathrm{d}^dx+gf^c_{ba}f^a p^{\alpha\mu}_c\mathrm{d}A^b_\alpha\wedge\mathrm{d}x_\mu\end{aligned} \quad (3.130)$$

The symmetrization and antisymmetrization of p occurs in the second and third terms respectively because of the symmetry of partial derivatives for the former and the anti-symmetry of the structure constant in the latter.

The right hand side, the gauge variation vector field contracted with the multisymplectic form, $X_{J_f}\lrcorner\,\Omega$:

$$X_{J_f}\lrcorner\,\Omega=X_{J_f}\lrcorner\,\mathrm{d}A^a_\alpha\wedge\mathrm{d}p^{\alpha\mu}_a\wedge\mathrm{d}x_\mu=(-D_\alpha f^a)\mathrm{d}p^{\alpha\mu}_a\wedge\mathrm{d}x_\mu-gf^c_{ba}f^a p^{\alpha\mu}_c\mathrm{d}A^b_\alpha\wedge\mathrm{d}x_\mu \quad (3.131)$$

This is equal to $\mathrm{d}J_f$ on the constaint surface $p^{(\alpha\beta)}_a=0$. Also,

$$X_{J_f}\lrcorner\,\Theta=(-D_\alpha f^a)\frac{\partial}{\partial A_\alpha}+gf^c(x)f^a_{bc}p^{\alpha\mu}_a\frac{\partial}{\partial p^{\alpha\mu}_b}\lrcorner\,(p^{\alpha\mu}_a\mathrm{d}A^a_\alpha\wedge\mathrm{d}x_\mu) \quad (3.132)$$

$$= (-D_\alpha f^a) p_a^{\alpha\mu} \mathrm{d}x_\mu \;=\; -J_f \tag{3.133}$$

So X_{J_f} is exact hamiltonian.

$$\begin{aligned}\mathcal{L}_{X_{J_f}} \Theta = X_{J_f} \lrcorner\, \mathrm{d}\Theta + \mathrm{d}(X_{J_f} \lrcorner\, \Theta) = -X_{J_f} \lrcorner\, \Omega + \mathrm{d}J_f \\ = -((\partial_\beta \partial_\alpha f^a) p_a^{(\alpha\beta)} \mathrm{d}^d x + (g f^c_{ba} A^b_\alpha \partial_\beta f^a) p_a^{\alpha\beta} \mathrm{d}^d x) \\ \stackrel{c}{=} \; -(g f^c_{ba} A^b_\alpha \partial_\beta f^a) p_a^{\alpha\beta} \mathrm{d}^d x\end{aligned} \tag{3.134}$$

The Lie algebra of gauge variations $\mathfrak{g}$ is non-abelian. An element of $\mathfrak{g}$ is the list of functions $(f^a(x))$ on spacetime.

The moment map is $\mathcal{M} \longrightarrow \mathfrak{g}* :: m = (A^a_\alpha, p_a^{\alpha\mu}, x) \mapsto \mathfrak{I}(m) = p_a^{(\alpha\mu)} \mathrm{d}x_\mu$

For $f(x) \in \mathfrak{g}$, $\langle \mathfrak{I}(m), f \rangle \;=\; (-D_\alpha f^a) p_a^{(\alpha\mu)} \mathrm{d}x_\mu$

The moment map is zero on the surface $p_a^{(\alpha\beta)} = 0$. Constraint surface is $M_G = \{(A^a_\alpha, p_a^{[\alpha\beta]}, x)\}$. We want to show that $Y \lrcorner\, \Omega_G = Y \lrcorner\, \mathrm{d}A^a_\alpha \wedge \mathrm{d}p_a^{[\alpha\mu]} \wedge \mathrm{d}x_\mu = 0$ implies $Y = -D_\alpha f^a(x) \frac{\partial}{\partial A^a_\alpha} - g f^c(x) f^c_{ba} p^{\alpha\mu} \frac{\partial}{\partial p^{\alpha\mu}}$.

The second stage in symplectic Marsden-Weinstein reduction is to mod-out the leaves of the characteristic distribution tangent to the constraint submanifold M_G. In the multisymplectic setting this is better viewed as mod-ing out functions of the leaves.

The Lie derivative of the multisymplectic form with respect to an arbitrary infinitesimal variation,

$$\delta = B^a_\alpha(x, A) \frac{\partial}{\partial A^a_\alpha} + C_b^{\alpha\mu}(x, p) \frac{\partial}{\partial p_b^{\alpha\mu}} \tag{3.135}$$

is:

$$\begin{aligned}\mathcal{L}_{X_\delta} \Omega = X_\delta \lrcorner\, \mathrm{d}\Omega + \mathrm{d}(X_\delta \lrcorner\, \Omega) = \mathrm{d}(B^a_\alpha \frac{\partial}{\partial A^a_\alpha} + C_b^{\alpha\mu} \frac{\partial}{\partial p_b^{\alpha\mu}} \lrcorner\, \mathrm{d}A_\alpha \wedge \mathrm{d}p^{\alpha\mu} \wedge \mathrm{d}x_\mu) = \\ (\partial_\mu B^a_\alpha)\, \mathrm{d}p_a^{\alpha\mu} \wedge \mathrm{d}^d x - (\partial_\nu C_b^{\alpha\mu}) \mathrm{d}A^a_\alpha \wedge \mathrm{d}^d x + (\partial^\nu_b B^a_\alpha) \mathrm{d}A^b_\nu \wedge \mathrm{d}p_a^{\alpha\mu} \wedge \mathrm{d}x_\mu \\ + (\partial^b_{\kappa\lambda} C_a^{\alpha\mu}) \mathrm{d}A^a_\alpha \wedge \mathrm{d}p_b^{\kappa\lambda} \wedge \mathrm{d}x_\mu\end{aligned} \tag{3.136}$$

where we used abbreviations $\partial^\nu_b := \frac{\partial}{\partial A^b_\nu}$ and $\partial^b_{\kappa\lambda} := \frac{\partial}{\partial p_b^{\kappa\lambda}}$.

We now also calculate the Lie derivative of the tautological form with respect to an arbitrary infinitesimal variation:

$$\mathcal{L}_{X_\delta} \Theta = X_\delta \lrcorner\, \Omega + \mathrm{d}(X_\delta \lrcorner\, \Theta) = (C_b^{\beta\mu} + p_a^{\alpha\mu} \partial^\beta_b B^a_\alpha) \mathrm{d}A^b_\beta \wedge \mathrm{d}x_\mu + \partial_\mu B^a_\alpha p_a^{\alpha\mu} \mathrm{d}^d x \tag{3.137}$$

This is not zero unless $B^a_\alpha =$ constant and $C_b^{\alpha\mu} = 0$, so this is not in general an exact multi-symplectomorphism over multiphase space. We want to restrict the multisymplectic form to the constraint surface $p_a^{(\alpha\beta)} = 0$ in multiphase space, and find the conditions on the coefficients B and C which result in $\mathcal{L}_{X_\delta}\Theta = 0$ on the constraint surface.

We require the components to be zero on the constraint surface $p_a^{(\alpha\beta)} = 0$:

$$C_b^{\beta\mu} + p_a^{[\alpha\mu]}\partial_b^\beta B_\alpha^a = 0 \quad \text{and} \quad \partial_{[\mu} B_{\alpha]}^a = 0 \tag{3.138}$$

These conditions and the functional dependence indicated in (3.135) mean that the functions B_α^a and $C_b^{\beta\mu}$ are of the form

$$B_\alpha^a = -\partial_\alpha f^a(x) - g f^a(x) f_{ba}^c A_\alpha^b \quad \text{and} \quad C_b^{\beta\mu} = g f^a(x) f_{ba}^c p_c^{[\beta\mu]} \tag{3.139}$$

where f_{ba}^c are constants. We also require

$$\partial_{[\mu} f^a(x) A_{\alpha]}^b f_{ba}^c \;= 0 \tag{3.140}$$

We want these conditions to produce the gauge variations $B = -D_\alpha f^a$ and $C = g f^a(x) f_{ba}^c p_c^{\alpha\mu}$.

We choose the temporal gauge $A_0^a = 0$, which is a pointwise constraint in multiphase space unlike the Lorenz or Coulomb gauges which are constraints which are spacetime partial derivatives of A. Then the reduced multisymplectic form is

$$\Omega_{GG} = \mathrm{d}A_i^a \wedge (\mathrm{d}p_a^{[ij]} \wedge \mathrm{d}x_j + \mathrm{d}p_a^{[i0]} \wedge \mathrm{d}x_0) = \mathrm{d}A_i^a \wedge (\mathrm{d}(\epsilon^{ijk} B_{ka}) \wedge \mathrm{d}x_j + \mathrm{d}E_a^i \wedge \mathrm{d}x_0) \tag{3.141}$$

3.8.4 The multiphase-space energy-momentum tensor

In section 3.4.4, the multimomentum energy momentum tensor density is defined to be:

$$\tilde{T^\mu}_\nu \;:=\; \frac{\partial\mathcal{H}}{\partial p_a^{\kappa\nu}} p_a^{\kappa\mu} - (p_a^{\kappa\lambda}\frac{\partial\mathcal{H}}{\partial p_a^{\kappa\lambda}} - \mathcal{H})\delta_\nu^\mu \;=\; \mathcal{U}_{\kappa\nu}^a p_a^{\kappa\mu} - \tilde{\mathcal{L}}_{MP}\,\delta_\nu^\mu \;=\; \mathcal{U}_{\kappa\nu'}^a p_a^{\kappa\mu'}\,\delta_{\nu\mu'}^{\mu\nu'} \,+\, \mathcal{H}\,\delta_\nu^\mu \tag{3.142}$$

where the the following are used: the strain observable $\mathcal{U}_{\kappa\nu}^a := \frac{\partial\mathcal{H}}{\partial p_a^{\kappa\nu}}$, the multiphase-space Lagrangian observable $\tilde{\mathcal{L}}_{MP} := p_a^{\kappa\nu}\frac{\partial\mathcal{H}}{\partial p_a^{\kappa\nu}} - \mathcal{H}$, and the antisymmetrizer $\delta_{\nu\mu'}^{\mu\nu'} := \delta_{\mu'}^{\mu}\delta_\nu^{\nu'} - \delta_\nu^\mu\delta_{\mu'}^{\nu'}$.

Substituting the DDW Hamiltonian 3.106 for the Yang-Mills field, which is:

$$\begin{aligned}
\mathcal{H} = p_a^{\mu\nu}\partial_\mu A_\nu^a \;-\mathcal{L} &= \frac{1}{4}p_a^{[\mu\nu]}p_{[\mu\nu]}^a - \frac{g}{2}p_a^{[\mu\nu]} f_{bc}^a A_\mu^b A_\nu^c + p_a^{(\mu\nu)}\partial_\mu A_\nu^a \\
&= \frac{1}{4}p_a^{[\mu\nu]}p_{[\mu\nu]}^a - \frac{g}{2}p_a^{[\mu\nu]} f_{bc}^a A_\mu^b A_\nu^c - \partial_\mu p_a^{(\mu\nu)} A_\nu^a \\
&\stackrel{c}{=} \frac{1}{4}p_a^{[\mu\nu]}p_{[\mu\nu]}^a - \frac{g}{2}p_a^{[\mu\nu]}[A_\mu, A_\nu]^a
\end{aligned} \tag{3.143}$$

we obtain the multiphase-space energy-momentum tensor,

$$\tilde{T^\mu}_\nu \;:= \frac{1}{2}p_{[\kappa\nu]}^a p_a^{\kappa\mu} - \frac{g}{2}p_a^{\kappa\mu} f_{bc}^a A_{[\kappa}^b A_{\nu]}^c + p_a^{\kappa\mu}\partial_{(\kappa}A_{\nu)}^a \;-(\frac{1}{4}p_a^{[\mu\nu]}p_{[\mu\nu]}^a)\delta_\nu^\mu \tag{3.144}$$

3.9 Summary

This chapter introduced the use of multimomenta in the study of fields and showed how similar notions to those in classical Hamiltonian mechanics are used to analyse fields. These were employed in the analysis of asymmetric system such as the Yang-Mills field above, and other fields, in less detail, in the appendix. They will be employed in the next chapter, on BRST, and in the subsequent chapters, for the sigma model of J-holomorphic curves. They will also be employed for the multiphase space Hamilton-Jacobi theory in the appendix.

Chapter 4

BRST

In this chapter, the BRST approach to Marsden-Weinstein reduction is described, firstly on phase space and then in the generalization to multiphase space. These are followed by the examples of the electromagnetic and Yang-Mills field theories, continuing on from the example in chapter 3.

4.1 Introduction

The BRST (named after Becchi, Rouet, Stora, Tyutin [80] [50]) formalism is a method employing homological algebra to obtain the reduced phase space from a phase space with constraints or symmetries, and is well suited for methods of canonical quantization and QFT. The pioneers of BRST were: Fadeev and Popov [64], invented ghosts fields for gauge fixing. Rouet, Stora, Becchi [80] and Tyutin [50] discovered the BRST supersymmetry and its properties. Kugo and Ojima [49] its relationship to canonical quantization and obtaining the correct Fock space.

BRST takes the viewpoint of algebraic geometry to characterize the rings of functions on the manifolds involved. In particular it employs resolutions of the ring of functions on the physical phase space into complexes. A resolution is an embedding of an algebraic object into a larger object which is easier to deal with, in this case a bigraded differential complex, which can be viewed as a super phase space with a (graded) Poisson bracket. It avoids dealing with the (often complicated) reduced phase space directly, and instead uses a relatively straightforward extension of the (usually simpler) phase space in which the symmetric system is defined, and employs a special observable in the graded phase space (the BRST charge corresponding to the differential of the BRST complex) to project out the physical subsystem (the physical observables as homology classes). The BRST formalism employs a Poisson algebra on a super phase-space extension of the phase space of a dynamical system, where the physical observables are obtained as observables which Poisson-commute with a special observable called the BRST charge, and which is readily quantized.

The physical reduced ring of observables (functions on the reduced phase space), which is the quotient of the ring of functions on the constraint surface in the original phase space by the orbits of the gauge group action, is obtained in the BRST method as a cohomology ring of functions on the BRST complex. This cohomological reduction is an algebraic implementation of Marsden-Weinstein reduction defined in section 2.4.2. The algebra of functions on the reduced symplectic manifold are the physical observables and are obtained in the BRST formalism as the zeroth cohomology ring in a graded complex with differential δ_B. The cohomology preserves Poisson structures, so the complex can be viewed as a super-phase-space enlargement of the original phase space (with grassmann odd ghosts and ghost momenta adjoined to the original phase-space coordinates, where the number of ghosts and ghost momenta factors in a term determine the bidegree (l, m) in the bicomplex). There are two commuting differentials d and δ (of degree 1 and -1) from which the BRST differential δ_B of the BRST complex is constructed: $\delta_B = \delta + (-1)^l \mathrm{d}$ in the most straightforward cases. The factor of (-1) ensures that $\delta_B^2 = 0$ if $\delta^2 = 0 = \mathrm{d}^2$. The use of supermanifolds here, that is, manifolds with some grassmann odd coordinates, is based on the work of DeWitt [16] and Rogers [9]. The algebra of functions on on super phase space has a (graded) Poisson bracket and there is a grassmann odd generator (the BRST observable) Q which generates the BRST variation $\delta_B \cdot = -\{Q, \cdot\}$, and which encodes the gauge action within the enlarged phase space. These additional (grassmann odd) degrees of freedom in enlarged phase space are the gauge variation parameters, so that the local gauge symmetry of the original Lagrangian becomes a global symmetry in the enlarged phase space, under the variation where the value of the parameters is the value of the additional grassmann odd degrees of freedom. This variation is called a BRST variation and a key problem is to construct the correct BRST variation of the additional grassmann odd degrees of freedom. The essential $\delta_B^2 = 0$ nilpotent property of the BRST differential, together with the graded Jacobi identity for the Poisson bracket, implies $2Q^2 = \{Q, Q\} = 0$, which is the key property of the BRST variation. This allows cohomology rings to be defined where $H^0_{\delta_B}$ will be the ring of gauge invariant functions on the constraint surface, the ring of physical observables. Gauge fixing is achieved by adding a suitable δ_B-exact gauge fixing term to the original δ_B invariant Hamiltonian or Lagrangian, which thereby remains δ_B invariant, thus obtaining gauge fixing without losing BRST symmetry.

Here we will assume that the gauge parameters, the generators of the Lie group, are grassmann even (commuting) quantities, and will correspond to odd (anticommuting) ghost degrees of freedom in the BRST construction. If the original gauge parameters are odd, then the corresponding ghosts would be grassmann even.

The BRST formalism can be used for grassmann odd, open, or reducible gauge parameters by employing even graded ghosts, the BV methodology, or ghosts for ghosts respectively. These will not be dealt with in this introduction. For open algebras and reducible gauge parameters, extra degrees of freedom, ghosts for ghosts, are adjoined to the model in the BV formalism, one for each relation between the ghosts and again of opposite parity. This can be iterated if there are relations among the ghost for ghosts. The BV formalism gives an explicit procedure to construct the BRST observable from the relations between gauge parameters, with successive terms in an expansion of the BRST observable $Q = Q_0 + Q_1 + Q_2 + \ldots$ constructed from the relations between the gauge generators in such a way as to ensure that $\{Q, Q\} = 0$.

Quantization

In canonical quantization of the BRST system, in suitably simple models, the differential complex and Lie structure carries over to the operator observables and to the Hilbert space, and this BRST model then has the properties of a supersymmetric model. The BRST method is designed so that the process of cohomological reduction and quantization commute, because the various mappings employed are Poisson maps. The BRST cohomological construction employs graded Poisson algebras, and the BRST complex can be readily quantized if the original phase space can be canonically quantized, and the physical observables and states will be the kernel of the BRST charge Q mod the image of Q, where Q has been promoted to a quantum operator on a (non-positive definite) 'Hilbert' space, and the super-Poisson bracket is promoted to a super-commutator. The odd canonial pairs in the super-phase space are the ghosts and ghost momenta in the Fade'ev-Popov quantization scheme. The Fade'ev-Popov ghosts are included because, in the path integral Lagrangian, gauge fixing by itself is usually not enough. This is due to the measure in the functional integral, which is the Jacobian of gauge transformations, of varying the gauge fixing surface, having to be taken into account, so that the functional integral is independent of the choice of gauge fixing. In the path integral, when the functional integral is performed over the ghost pairs $(c, \bar{c})$, a term like $\bar{c}\partial Dc$ in the Lagrangian becomes a functional determinant $\det(D)$ factor in the functional integral, which cancels the functional determinant due to the gauge degrees of freedom, which arises from the Jacobian of the embedding of the gauge fixing surface in phase space. This makes the path integral invariant under gauge variations of the gauge fixing, which is the requirement that is aimed at. The cancellation occurs exactly by the BRST construction because the ghost extended Lagrangian, including the gauge fixing, is BRST symmetric, because the gauge symmetry has been extended to the gauge fixing term by the use of a BRST-exact gauge fixing term mentioned above.

Complications in BRST quantization are non-zero higher cohomology classes and the treatment of non-positive-definite norms on states. The Gribov ambiguity [99], where a global gauge fixing term may not exist (because the Dirac bracket is not defined at every point on the reduced phase space), may still be present - but in the BRST approach the path integral is still well defined and can still be calculated so long as the gauge fixing fermion obeys certain non-singularity conditions [7]. The existence of the reduced phase space is a separate issue from the existence of a global section (i.e. a gauge fixing function). The space of (physical) observables on the reduced phase space is isomorphic to the zeroth Q-cohomology group of functions on the BRST enlarged phase space and this can be quantized so that the (Hilbert) space of physical states is isomorphic to the zeroth Q-cohomology group of states on the BRST enlarged (Krein) space space of states [85]. It can be shown that if the gauge-fixing fermion Ψ is chosen so that $[Q, \Psi]$ satisfies certain 'non-singularity' conditions, the supertrace of the evolution operator constructed from the BRST gauge-fixed Hamiltonian is equal to the trace over physical states of the evolution operator constructed from the original Hamiltonian [7]. Although the terms 'gauge-fixing function' and 'gauge-fixing fermion' are similar, the roles they play, the former in the Dirac bracket and the latter in BRST, are different.

4.2 The algebraic BRST bigraded complex

The BRST complex is obtained from a bigraded differential complex which is a resolution of the gauge action (the Chevalley-Eilenberg complex of a Lie algebra module [26]) on the Koszul resolution of the algebra of a regular ideal of constraint functions in phase space in terms of free modules. The definition of a *regularity* for a set of constraint functions can be made in various equivalent ways: one definition is, if $T^a(z^\alpha) = 0$ with $a = 1 \ldots K \leq N$ and $\alpha = 1 \ldots 2N$ are constraint equations then the matrix (with row index a and column index α) $\left[\frac{\partial T^a}{\partial z^\alpha}\right]$ is of maximum rank K at every point of the constraint surface. These two complexes, the Koszul and the Chevalley-Eilenberg, embody the two steps of Marsden-Weinstein reduction: from the phase space to the constrained phase space and then to the space of gauge orbits on the constrained phase space.

The Koszul complex is constructed from the constraint functions and the Chevalley-Eilenberg complex is constructed from the Lie algebra action on functions on phase space generated by the constraints. The two complexes are compatible because of this link between the Lie algebra and the constraints via the Poisson bracket. The sources for the next two sections is [53] [59] [62] [58] [78] [77] [4] [26]. Because of the algebraic nature of BRST, it is first worthwhile to start with an algebraic definition of Marsden-Weinstein reduction.

4.2.1 Coisotropic Marsden-Weinstein reduction expressed in terms of Poisson algebras

The BRST construction is an algebraic homological construction involving Poisson algebras of observables (functions of phase or super-phase space). Therefore it convenient to start by expressing Marsden-Weinstein reduction in terms of Poisson algebras. We are looking to an algebraic geometry definition of Marsden-Weinstein reduction in terms of the algebra of functions on the manifolds rather than the manifolds themselves.

We first note that if $\mathbb{M}$ is a symplectic manifold then $C^\infty(\mathbb{M})$ is a Poisson algebra with the Poisson bracket of functions (observables) defined from the symplectic form on $\mathbb{M}$. An embedded closed submanifold M_T of any manifold $\mathbb{M}$ defines an ideal I of functions of the manifold which are zero on that submanifold and there is a natural isomorphism $C^\infty(\mathbb{M}_T) \cong C^\infty(\mathbb{M})/I$. Let $\mathbb{M}_T$ be a coisotropic submanifold of a Poisson manifold $\mathbb{M}$, I is the vanishing ideal of $\mathbb{M}_T \subset \mathbb{M}$: the ring of functions on $\mathbb{M}$ which are zero on $\mathbb{M}_T$. If $\langle T\rangle$ is the ideal generated by a set of regular constraints $T_a = 0$ which define M_T, then $I = \langle T\rangle$. If T are first class constraints, $\{T_a, T_b\} \in \langle T\rangle \ \forall\ T_a, T_b$, then $\langle T\rangle$ is a Poisson sub-algebra of $C^\infty(\mathbb{M})$, but not necessarily a Poisson ideal in $C^\infty(\mathbb{M})$. (An ideal I_p is a Poisson ideal if it has the property $\{C^\infty(\mathbb{M}), I_p\} \subset I_p$). Thus the ring of functions on the constraint surface (which is the first step of Marsden-Weinstein reduction) is

$$C^\infty(\mathbb{M}_T) \cong C^\infty(\mathbb{M})/I \tag{4.1}$$

The coisotropy of $\mathbb{M}_T$ results in that I is a Poisson sub algebra of $C^\infty(\mathbb{M})$, called a coisotropic ideal. The above, $C^\infty(\mathbb{M}_T)$, is not a Poisson algebra because I is not a Poisson ideal of $C^\infty(\mathbb{M})$. However I is a Poisson ideal of the Poisson normalizer $N(I)$ of I in $C^\infty(\mathbb{M})$, where $N(I) := \{f \in C^\infty(\mathbb{M}) | \{f, I\} \subset I\}$, the Poisson subalgebra of $C^\infty(\mathbb{M})$ of functions which are constant on the orbits on $\mathbb{M}_T$ generated by $\{I, \cdot\}$ and therefore the quotient

$$N(I)/I \quad \cong \quad C^\infty(\mathbb{M}_{TT}) \tag{4.2}$$

is also a Poisson algebra, the reduced Poisson algebra of $C^\infty(\mathbb{M})$ by I , and is isomorphic to the Poisson algebra of functions $C^\infty(\mathbb{M}_{TT})$ on the reduced phase space $\mathbb{M}_{TT}$. This, $C^\infty(\mathbb{M})//I := N(I)/I$, is the Marsden-Weinstein reduction of $C^\infty(\mathbb{M})$. $R//I := N(I)/I$ can be taken to be the definition of algebraic Marsden-Weinstein reduction for a general Poisson algebra R relative to an ideal I.

Example: The following is a trivial example: $\mathbb{M} = \{(q^\mu, p_\mu)\}$, $T = p_0$, $I = \langle T\rangle = \langle p_0\rangle = \{\oplus_{k>0}\ p_0^k f_k(q^\mu, p_i)\ \}$ and $\mu = 0, \ldots, N : i = 1, \ldots N$.

$f_k(q^\mu, p_i)$ are arbitrary smooth functions of the coordinates indicated in the brackets (where, of course, the indices range over the values $\mu = 0, \ldots, N : i = 1, \ldots N$). This is a Poisson subalgebra because $\{p_0^k f_1(q^\mu, p_i), p_0^l f_2(q^\mu, p_i)\} = \oplus_{k>0} p_0^k f_k(q^\mu, p_i)$ but not an ideal because $\{p_0, q^0\} = 1 \notin I$. The Poisson normalizer $N(I)$ of I in $\mathbb{M}$ is $N(I) := \{f \in C^\infty(\mathbb{M}) | \{f, I\} \subset I\} = \{f(q^i, p_i)\}$, so we remove the functions dependent on q^0 to make the normalizer. $C^\infty(\mathbb{M}_T) \cong C^\infty(\mathbb{M})/I = \{f(q^\mu, p_i)\}$ mods out the functional dependence on p_0, and this is a sub algebra because $\{f_1(q^\mu, p_i), f_2(q^\mu, p_i)\} = f_3(q^\mu, p_i)$, but not an ideal because $\{f_1(q^\mu, p_\mu), f_2(q^\mu, p_i)\} = f_3(q^\mu, p_\mu)$ which may not belong to $C^\infty(\mathbb{M}_T)$. Clearly if the functional dependence on q^0 is removed then we have $N(I)/I = \{f(q^i, p_\mu)\}/\{\oplus_{k>0} p_0^k f_k(q^\mu, p_i)\} = \{f(q^i, p_i)\}$. As expected, the constraint $p_0 = 0$, leads to a reduced phase space $\{(q^i, p_i)\}$, where both the p_0 coordinate and it's canonical conjugate q^0 have been eliminated. In the algebraic approach here, this is expressed as the ring of functions $\{f(q^i, p_i)\}$ on the reduced phase space.

4.2.2 The Koszul resolution of the constraint submanifold

Constructing the Koszul differential complex is the first step of the Marsden-Weinstein symplectic reduction. The constraint surface in phase space is obtained in the form of the ring of functions on the constraint surface in phase space. This appears as the zeroth homology ring of the Koszul complex. The Koszul complex is the ring of functions on a larger space constructed by adjoining some extra (grassmann odd) coordinated to the phase space. The Koszul complex also has a grading and a differential associated with it.

The Koszul complex is a graded differential complex which is a free resolution of an R-module R/I where R is a ring with identity element (for us the ring of observables $C^\infty(\mathbb{M})$) and I is the ideal generated by a chosen sequence of elements of the ring (for us the constraints $T_a, a = 1 \ldots K$). Note that for us the ring of equivalence classes R/I is isomorphic to the ring of smooth functions on the constraint surface defined by the regular constraints $T_a = 0$ in $\mathbb{M}$, and this will turn out to be the zeroth homology ring of the Koszul complex.

First we have to define the algebraic condition equivalent to the functional regularity condition of the set of constraint functions $\{T_a\}$ on the constraint surface $T_a = 0$. A definition equivalent to the previously given functional regularity condition of the set of constraint functions is that the gradients ∇T_a of the constraint functions are linearly independent on the constraint surface, where ∇ is the set of partial derivatives with respect to local coordinates.

A sequence of elements $T_1, T_2, \ldots, T_K$ of a ring R is called regular if, for all L, and every ideal $I_L = R\langle T_1, T_2, \ldots, T_L\rangle$ generated by $T_1, T_2, \ldots, T_L$, any product of the form $ST_{L+1} = 0$ mod I_L implies $S = 0 \mod I_L$.

The Koszul complex constructed below is a Koszul resolution when the homology groups are zero for positive homology, which occurs when the sequence of elements is regular (which means that the set of constraints T is regular, which is the same as saying an irreducible set of constraints such that 0 is a regular value of T). A regular ideal is also known as a Borel ideal. Tate [60] showed how to resolve a non-regular ideal or a reducible set of constraints by employing a Koszul-Tate resolution.

Explicitly construction of the Koszul complex

The Koszul complex is the free R-module $K_T := R\langle\rho_m\rangle$ where the basis is

$$\{\rho_{|m_l|} := \rho_{m_1}\rho_{m_2}\dots\rho_{m_l}\ \forall m_1 < m_2 < \dots\langle m_l,\ \forall l = 1,\dots,K \text{ and } \rho_{|m_0|} := 1\} \tag{4.3}$$

where the set of generators $P = \{\rho_a\ ,\ a = 1\dots K\ ;\ \rho_0 = 1\ \}$ is a set of extra odd (i.e. anticommuting (except for $\rho_0 := 1$): $\rho_a\rho_b = -\rho_b\rho_a$) elements, which commute with the elements of R. $|m_l|$ is a multi-index which is shorthand for a product of l indexed elements and $\rho_{|m_0|} = \rho_{m_0} = \rho_0 := 1$. The degree l space K_l in the complex K_T has basis the $\binom{l}{K}$ products of l distinct odd elements from P: $\{\rho_{|m_l|} = \rho_{m_1}\rho_{m_2}\dots\rho_{m_l}, \forall m_1 < m_2 < \dots < m_l\}$. The degree 0 space is R. For phase-space BRST, $R = C^\infty(\mathbb{M})$, the ring of observables on phase space $\mathbb{M}$.

The differential, δ, of the Koszul complex is defined by $\delta f = 0$, for $f \in C^\infty(\mathbb{M})$, and $\delta\rho_a = T_a$ extended to K_T as an odd graded $\mathbb{R}$-linear derivation, which forces the differential to be nilpotent: $\delta\delta = 0$, as shown below, and reduces the degree l by one:

$$0 \xrightarrow{\delta_{K+1}} K^K \xrightarrow{\delta_K} K^{K-1} \xrightarrow{\delta_{K-1}} \dots \xrightarrow{\delta_3} K^2 \xrightarrow{\delta_2} K^1 \xrightarrow{\delta_1} K^0 \xrightarrow{\delta_0} 0 \tag{4.4}$$

Koszul homology

The higher homologies measure some (largely unknown) algebraic relations on the elements T_i. If these elements form a regular sequence then the higher homology rings are zero.

We will now calculate the homology rings:

For $2 \leq l \leq K$,

The image of the differential is: $\mathrm{im}\ (\delta_\mathrm{l}\ ) = \mathrm{R}\langle\ \sum_{\mathrm{a}=1}^{\mathrm{l}}\ (-1)^{\mathrm{a}-1}\ \mathrm{T}_{\mathrm{m_a}}\ \rho_{\mathrm{m_1}}\rho_{\mathrm{m_2}}\hat{\rho}_{\mathrm{m_a}}\dots\rho_{\mathrm{m_l}}\rangle$, where the hat notation $\hat{\rho}_i$ indicates that this factor is omitted from the product.

The kernel of the differential is: $\ker\ (\delta_{\mathrm{l}-1}\ ) = \mathrm{R}\langle\ \sum_{\mathrm{a}=1}^{\mathrm{l}}\ (-1)^{\mathrm{a}-1}\ \mathrm{T}_{\mathrm{m_a}}\ \rho_{\mathrm{m_1}}\rho_{\mathrm{m_2}}\hat{\rho}_{\mathrm{m_a}}\dots\rho_{\mathrm{m_l}}\rangle$, if the the sequence T is regular in R. And therefore, for $2 \leq l \leq K$ the positive degree homology rings of the Koszul complex are: $H_K^{l-1} = \frac{\ker\ (\delta_{l-1}\ )}{\mathrm{im}\ (\delta_l\ )} = 0$.

For l $= 1$,

im $(\delta_1\ ) = \mathrm{R}\langle\mathrm{T_a}\rangle = \mathrm{I}$, which is the ideal generated by $T_a, a = 1 \dots K$. For $l = 0$, im $(\delta_0\ ) = 0$ and so ker $(\delta_0\ ) = \mathrm{R} = \mathrm{K}^0$ and therefore the zeroth homology ring of the Kozul complex is $H_K^0 = \frac{\ker\ (\delta_0\ )}{\mathrm{im}\ (\delta_1\ )} = R/I$.

If R is the ring $C^\infty(\mathbb{M})$ of smooth functions on $\mathbb{M}$, and I the ideal $R\langle T_a\rangle$ generated by the regular constraints $T_a \in C^\infty(\mathbb{M})$, then the R-module R/I is isomorphic to the ring of smooth functions on the constraint surface $\mathbb{M}_T$ defined by $T_a = 0$ in $\mathbb{M}$, which is what the Koszul construction above represents with the homology ring. This completes the first step of the BRST approach of the Marsden-Weinstein reduction.

The Koszul complex is a projective resolution of R/I in the regular case (as defined above) where the only non-zero homology ring is H^0. The Koszul sequence augmented by the zeroth homology is a long exact sequence, a projective resolution of the module R/I in terms of free R-modules:

$$0 \longrightarrow K^K \xrightarrow{\delta_K} K^{K-1} \xrightarrow{\delta_{K-1}} \dots \xrightarrow{\delta_3} K^2 \xrightarrow{\delta_2} K^1 \xrightarrow{\delta_1} R \xrightarrow{\epsilon} R/I \longrightarrow 0. \tag{4.5}$$

where ϵ is the projection. This is called the Koszul resolution. The term 'resolution' refers to the fact that the the ring of functions on the constraint surface has been embedded in a larger algebra.

As will be seen below, the ρ_a will be the grassmann odd momenta coordinates of the BRST super-phase space.

It will be seen that the generating elements ρ_a will be have role as the basis elements of the symmetry group Lie algebra $\mathfrak{g}$ and will transform as such - as can be seen in the next subsection, in 4.2.3, in the definition of the Chavalley-Eilenberg direction of the bicomplex. In that section, a $\mathfrak{g}$ action will be then defined on the Koszul complex K_T which will then allow the Chevalley-Eilenberg complex $CE^\bullet(\mathfrak{g}, K_T)$ to be constructed. The Chevalley-Eilenberg complex will be the space of observables on BRST super phase space.

4.2.3 The Chevalley-Eilenberg complex

The construction of the Chevalley-Eilenberg complex is the second step of the Marsden-Weinstein symplectic reduction in BRST, where we seek the space of leaves into which the constraint surface has been foliated by the action of a Lie group G (which we assume to be free and proper): the set $\mathbb{M}_T/G$ of orbits $G \cdot x$, generated by the constraint functions, on the constraint submanifold $\mathbb{M}_T$ in phase space $\mathbb{M}$. Each leaf has a G action. Under the conditions that the action is free and proper, the reduced space $\mathbb{M}_{TG} := \mathbb{M}_T/G$ is a manifold, which we assume here.

General Chevalley-Eilenberg complex

The Chevalley-Eilenberg complex $CE^\bullet(\mathfrak{g}; \mathfrak{M})$ is a graded differential complex constructed from a Lie algebra $\mathfrak{g}$-module $\mathfrak{M}$ over a Lie algebra $\mathfrak{g}$ over a field $\mathbb{K}$. The cohomology, called the *Lie algebra cohomology*, encodes information about the Lie algebra $\mathfrak{g}$ and the representation on $\mathfrak{M}$. In particular the zeroth cohomology $H^0(\mathfrak{g}, \mathfrak{M})$ is the ring of invariants $\mathfrak{M}^\mathfrak{g} \subset \mathfrak{M}$ where $\mathfrak{g} \cdot \mathfrak{M}^\mathfrak{g} = 0$. The Lie algebra $\mathfrak{g}$ is semisimple iff $H^0(\mathfrak{g}, \mathfrak{M}) = 0$ for all finite-dimensional modules $\mathfrak{M}$. Of particular use is the trivial representation $\mathfrak{M} = \mathbb{K}$, where $H^1(\mathfrak{g}, \mathbb{K}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}]$, the abelianization of $\mathfrak{g}$, and $H^2(\mathfrak{g}, \mathbb{K})$ is isomorphic to the space of equivalence classes of central extensions. If $\mathfrak{g}$ is semisimple then $H^1(\mathfrak{g}, \mathbb{K}) = \mathfrak{g}/[\mathfrak{g}, \mathfrak{g}] = 0 = H^2(\mathfrak{g}, \mathbb{K})$. Another example is the Chevalley-Eilenberg complex of the adjoint representation $\mathfrak{M} = \mathfrak{g}$, where the cohomology ring $H^0(\mathfrak{g}, \mathfrak{g})$ is the center of $\mathfrak{g}$, $H^1(\mathfrak{g}, \mathfrak{g})$ is isomorphic to the space of outer derivations, and $H^2(\mathfrak{g}, \mathfrak{g})$ is isomorphic to the space of infinitesimal deformations. In BRST, $\mathfrak{M}$ will be in the first instance the ring of observables on the constraint surface, $C^\infty(\mathbb{M}_T)$, where the $\mathfrak{g}$ action will be the Poisson hamiltonian symplectomorphisms corresponding to the constraints T_a, and then, in the second instance, K_T, which will be turned into a $\mathfrak{g}$-module with a natural adjoint $\mathfrak{g}$-action on the Koszul generators ρ_a which are now treated as basis elements of the Lie algebra. This Chevalley-Eilenberg complex of the Koszul complex K_T will be our bigraded complex from which the BRST complex will be defined.

The Chevalley-Eilenberg complex is the exterior algebra over $\mathfrak{g}^*$ with coefficients in $\mathfrak{M}$, where the product of basis elements $c^a \in \mathfrak{g}^*$ is free and anti-commutative:

$$CE^\bullet(\mathfrak{g}; \mathfrak{M}) = \bigoplus_{l=0}^{\dim \mathfrak{g}} CE^l(\mathfrak{g}; \mathfrak{M}) \cong \Lambda^\bullet(\mathfrak{g}^*) \otimes \mathfrak{M}, \tag{4.6}$$

$$\text{where } CE^l(\mathfrak{g}; \mathfrak{M}) := \mathrm{Hom}_\mathbb{K}(\Lambda^l(\mathfrak{g}), \mathfrak{M}) \cong \Lambda^l(\mathfrak{g}^*) \otimes \mathfrak{M} \text{ and } CE^0(\mathfrak{g}; \mathfrak{M}) := \mathfrak{M}. \tag{4.7}$$

The grassmann odd basis elements c^a of $\mathfrak{g}^*$ are known as Fade'ev-Popov ghosts. The $\mathbb{Z}$ grading 'ghost degree' of the subspaces $CE^l(\mathfrak{g}; \mathfrak{M})$ is given by $l = 0 \ldots \dim \mathfrak{g}$. The grade degree $+1$ coboundary operator d is defined on $f \in \mathfrak{M}$ by the action of $\mathfrak{g}$ on $\mathfrak{M}$: $\mathrm{d}f = c_f \in \Lambda^1(\mathfrak{g}^*) \otimes \mathfrak{M}$, where c_f is defined so that $\langle c_f, X \rangle = X \cdot f$, where $X \in \mathfrak{g}$, where $\langle c^a, X_b \rangle = \delta^a_b$ is the dual pairing of basis elements of $\mathfrak{g}^*$ and $\mathfrak{g}$. The coboundary operator d is defined on $c \in \mathfrak{g}^*$ via the Lie algebra bracket: $(\mathrm{d}c)(X, Y) = -\langle c, [X, Y] \rangle$. This is extended to the whole of $\Lambda^\bullet(\mathfrak{g}^*) \otimes \mathfrak{M}$ as a graded derivation. From this definition and the Jacobi identity of the Lie algebra bracket, as well as the commutator identity (Lie module property) for the Lie algebra action of $\mathfrak{g}$ on $\mathfrak{M}$, the differential is nilpotent: $\mathrm{d}^2 = 0$, making $\Lambda^\bullet(\mathfrak{g}^*) \otimes \mathfrak{M}$ into a complex, the *Chevalley-Eilenberg complex*. As a result, we can define the cohomology ring $H^\bullet(\mathfrak{g};\mathfrak{M})$, called the Lie algebra cohomology with coefficients in $\mathfrak{M}$. Note that $H^0(\mathfrak{g};\mathfrak{M}) = \ker(\mathfrak{g}) = \mathfrak{M}^{\mathfrak{g}}$, the $\mathfrak{g}$ invariants of $\mathfrak{M}$. The differential d can be expressed with dual pair sets of basis elements of the Lie algebra, $\langle c^a, X_b\rangle = \delta^a_b$: $\mathrm{d} = c^a \wedge X_a \cdot -\frac{1}{2} c^a \wedge c^b \wedge f^c_{ab} X_c \lrcorner$,where f^c_{ab} are the structure constants of the Lie algebra $\mathfrak{g}$ with basis $\{X_a\}$. $\Lambda^\bullet(\mathfrak{g}^*) \otimes \mathfrak{M}$ is also a $\mathfrak{g}$-module: the grade-degree l spaces $\Lambda^l(\mathfrak{g}^*) \otimes \mathfrak{M}$ have a natural $\mathfrak{g}$-module structure where the $\mathfrak{g}$-module action is extended to the exterior products of the dual Lie algebra as the exterior power of the coadjoint representation and can be written explicitly as:

$$X \cdot \omega = \mathcal{L}_X \omega := [i_X, d]_+ \omega = (i_X \mathrm{d} + \mathrm{d} i_X)\omega \tag{4.8}$$

If $\mathrm{d}\omega = 0$ and so ω is a cochain, then the above is $X \cdot \omega = \mathrm{d} i_X \omega$ is exact and so the $\mathfrak{g}$-module action commutes with the differential. This results in the $\mathfrak{g}$-module action being trivial on the cohomology.

A formula for the Chevalley-Eilenberg differential acting on degree l element ω of $\Lambda^\bullet(\mathfrak{g}^*) \otimes \mathfrak{M}$ and then contracting with $l+1$ Lie algebra elements $X_1 \wedge \cdots \wedge X_{l+1}$ is:

$$\begin{aligned}(\mathrm{d}\omega)(X_1 \wedge \cdots \wedge X_{l+1}) &= \omega(\sum_{i<j}^{l+1} (-1)^{i+j}([X_i, X_j] \wedge X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge \hat{X}_j \wedge \ldots \wedge X_{l+1})) \\ &\quad - \sum_{i=1}^{l+1} (-1)^i X_i \cdot (\omega(X_1 \wedge \ldots \wedge \hat{X}_i \wedge \ldots \wedge X_{l+1})). \end{aligned} \tag{4.9}$$

Example: Vector fields on a manifold

Note that, for the example of the Lie algebra of vector fields on a manifold, this is just a definition of the deRham differential on forms and the action of the Lie algebra is the derivation action of vector fields on functions and the Lie derivative of forms on the manifold: $CE^\bullet(\mathfrak{X}(\mathbb{M}), C^\infty(\mathbb{M})) = (\Omega^\bullet(\mathbb{M}), \mathrm{d}_{deRham})$, where $\mathfrak{X}(\mathbb{M})$ is the Lie algebra of smooth vector fields on a manifold $\mathbb{M}$. The nilpotence property $\mathrm{d}^2 = 0$ of the differential can be seen to depend of the Jacobi identity for the Lie brackets of vector fields [,], as well as the assumed Poisson property of the representation of the Lie algebra on functions which in the deRham case follows from the fact that it is a defining representation.

Example: The adjoint representation

The CE complex is $CE^\bullet(\mathfrak{g};\mathfrak{g})$. The Lie algebra $\mathfrak{g}$ is itself a $\mathfrak{g}$ -module where the action is the Lie algebra bracket: $X_a \cdot X_b := [X_a, X_b]$. The Jacobi identity ensures the Lie module property, $[X_a, X_b] \cdot X_c := [X_a, X_b \cdot X_c] - [X_b, X_a \cdot X_c]$.

The cohomology of the CE complex is $CE^\bullet(\mathfrak{g};\mathfrak{g})$ is isomorphic to $CE^\bullet(\mathfrak{X}_R(G);C^\infty(G))$, where $\mathfrak{X}_R(\mathbb{M})$ is the Lie algebra of right invariant vector fields on G and $G = \exp(\mathfrak{g})$ the connected Lie group for the Lie algebra $\mathfrak{g}$. The differential d_G here is the deRham exterior derivative d on the manifold G.

We also have $CE^\bullet(\mathfrak{g};\mathfrak{g}) \cong (\Omega^\bullet_R(G), \mathrm{d}_{deRham})$, where $\Omega^\bullet_R(G)$ is the exterior algebra of right invariant forms on G. Note that $\Omega^1_R(G) \simeq \mathfrak{g}^*$

The Chevalley-Eilenberg complex on the $\mathfrak{g}$-module $C^\infty(\mathbb{M}_T)$

We now apply the CE construction to the ring of functions $C^\infty(\mathbb{M}_T)$ on the constraint surface $\mathbb{M}_T$ to define $CE^\bullet(\mathfrak{g};C^\infty(\mathbb{M}_T))$.

We define basis 1-form fields c^a on $\mathbb{M}_T$ dual to the basis vector fields X_a of the Lie algebra action of $\mathfrak{g}$ on $\mathbb{M}_T$: $X_a \lrcorner c^b = \delta^b_a$. Because the dimensionality of the X_a, which span the tangent space of the leaf (which is the orbit of G through m) through each point m in $\mathbb{M}_T$, is less than that of $\mathfrak{T}_m\mathbb{M}_T$, there is some arbitrariness in the choice of the basis 1-forms fields c^a's. This set can be viewed as a vector-valued 1-form which defines a connection on the bundle $\pi : \mathbb{M}_T \longrightarrow \mathbb{M}_{TG}$, where $\mathbb{M}_{TG}$ is the space of leaves. We then define a leaf (fibers on this fiber bundle) exterior derivative d_G on the exterior algebra of forms $\Omega^\bullet_G(\mathbb{M}_T)$ on $\mathbb{M}_T$ generated by the 1-forms c^a as follows:

$$\mathrm{d}_G f = X_a(f)c^a \tag{4.10}$$

$$\mathrm{d}_G c^a = -\frac{1}{2} f^a_{bc} c^b c^c \tag{4.11}$$

extended to all of $\Omega^\bullet_G(\mathbb{M}_T)$ as a graded derivation (a term is graded by the number of factors c^a). $f^a_{bc} \in C^\infty(\mathbb{M})$ are the structure functions of the X_a's under the commutator bracket of vector fields : $[X_a, X_b] = f^k_{ab}X_k$. d_G is the exterior derivative d on the manifold $\mathbb{M}_T$, projected down to the leaf tangent space. The above can be viewed as the structure equations for basis forms c^a. Therefore $\mathrm{d}^2_G = 0$,which defines a complex with differential d_G which is the Chevalley-Eilenberg complex $CE_{\mathbb{M}_T}$ of the Lie algebra $\mathfrak{g}$-module $C^\infty(\mathbb{M}_T)$.

The zeroth cohomology ring is $H^0_{CE} \simeq C^\infty(\mathbb{M}^{\mathfrak{g}}_T) \simeq C^\infty(\mathbb{M}_{TG})$, the ring of physical observables, which is our objective. This is clear from the fact that functions which are constant on the leaves form the kernel of the leaf derivative d_G, from the fact that the latter sees only change in the leaf directions in functions on $\mathbb{M}_G$: $H^0_{CE} = \frac{\ker\ (\mathrm{d}_1\ )}{\mathrm{im}\ (\mathrm{d}_0\ )} = \frac{C^\infty(\mathbb{M}_{TG})}{\{0\}} = C^\infty(\mathbb{M}_{TG})$. The 1-forms c^a are the Fade'ev-Popov ghosts and can be considered as the grassman odd generators of a vector space over $C^\infty(\mathbb{M}_T)$, where graded multiplication with other c^a's is simply the wedge product of 1-forms. The c^a can be viewed as a basis of the vector space dual, $\mathfrak{g}^*$, of the Lie algebra of the group action: $\langle c^a, X_b\rangle = \delta^a_b$.

4.2.4 Defining the Chevalley-Eilenberg complex to the Koszul complex K_T

Whereas we have only obtained $CE^\bullet(\mathfrak{g}; C^\infty(\mathbb{M}_T))$ so far, we now want to work with the algebra of functions over the the original phase space $\mathbb{M}$ rather than over the constraint surface $\mathbb{M}_T$. We know that $C^\infty(\mathbb{M})$ is the zeroth degree space of the Koszul complex K_T defined above, from which we can retrieve $C^\infty(\mathbb{M}_T)$ as a homology ring. So we will define $CE^\bullet(\mathfrak{g}; K_T)$.

We will now extend the Chevalley-Eilenberg complex over $C^\infty(\mathbb{M}_T)$, $CE_{M_T} := CE^\bullet(\mathfrak{g}; C^\infty(\mathbb{M}_T))$, to a Chevalley-Eilenberg complex, $CE_{K_T} := CE^\bullet(\mathfrak{g}; K_T) \cong \Lambda^\bullet(\mathfrak{g}^*) \otimes \Lambda^\bullet(\mathfrak{g}) \otimes C^\infty(\mathbb{M})$, over the Koszul complex K_T. Here we have identified the space over $\mathbb{R}$ of the free generators ρ_a of the Koszul complex with the Lie algebra $\mathfrak{g}$. As shown above in section 4.2.1, the Koszul complex K_T, which is constructed from $\mathbb{M}$, embodies $\mathbb{M}_T$ as its zeroth cohomology group. The Chevalley-Eilenberg complex, $CE_{K_T} \cong \bigoplus_{k,l=0}^{\dim \mathfrak{g}} \Lambda^k(\mathfrak{g}) \otimes \Lambda^l(\mathfrak{g}^*) \otimes C^\infty(\mathbb{M})$ is constructed as a bicomplex with grading $(k,l); k,l = 0,\ldots,K$, with two differentials δ and d_G with grade (-1 , 0), (0,1) respectively, and which commute with each other: $\delta \mathrm{d}_G = \mathrm{d}_G \delta$.

We consider the bi-graded space $CE_{K_T} = \Lambda^\bullet(\mathfrak{g}) \otimes \Lambda^\bullet(\mathfrak{g}^*) \otimes C^\infty(\mathbb{M})$, where $\Lambda^\bullet(\mathfrak{g}^*)$ is the exterior algebra of the group-orbit-leaf one-form fields viewed as the the basis of the dual of the Lie algebra (the ghosts), and $\Lambda^\bullet(\mathfrak{g})$ as the exterior algebra of the Koszul generators viewed as the basis of the Lie algebra (the ghost momenta). This is the set of polynomials in ghosts and ghost momenta with coefficients in $C^\infty(\mathbb{M})$, and this will be the ring of observables on the BRST super-phase space.

We note that the Koszul complex $K_T = \Lambda^\bullet(\mathfrak{g}) \otimes C^\infty(\mathbb{M})$ is a $\mathfrak{g}$-module with action $\rho_a \cdot f = -\{T_a, f\}$, where $f \in C^\infty(\mathbb{M})$, and $\rho_a \cdot \rho_b = [\rho_a, \rho_b] = -\{T_a, T_b\}^l \rho_l$, extended to K_T as a graded derivation. Because K_T is a $\mathfrak{g}$-module, we can define a Lie algebra Chevalley-Eilenberg complex $CE_{K_T} := CE^\bullet(\mathfrak{g}, K_T)$. We first extend the Koszul differential complex K_T to a differential complex over CE_{K_T} by defining the action of the Koszul differential δ on the C-E generators c^a as:

$$\delta c^a = 0. \tag{4.12}$$

We also need to define extend the action of the leaf derivation d_G on functions on $C^\infty(\mathbb{M}_T)$ to an action of the derivation d_G on $C^\infty(\mathbb{M})$:

$$\mathrm{d}_G f = -\{T_a, f\} c^a = X_a(f) c^a. \tag{4.13}$$

$$\mathrm{d}_G c^a = -\frac{1}{2} f^a_{bc} c^b c^c \tag{4.14}$$

where X_a, f, c^a, and f^a_{bc} are defined in the same way as before on all the leaves of the foliation of G-orbits in $\mathbb{M}$ instead of just in $\mathbb{M}_T$. We also need to define d_G on the Koszul generators

ρ_a as:

$$\mathrm{d}_G \rho_b = -f^a_{bc} \rho_a c^c = f^a_{bc} c^c \rho_a. \tag{4.15}$$

d_G is then extended to the whole of CE_{K_T} as a graded derivation.

It can be seen from the above tensorial index notation that the Koszul grassmann-odd generators, $\rho_a \in \mathfrak{g}$, transform as Lie algebra basis elements, as mentioned above, while the Chevalley-Eilenberg grassmann-odd generators, $c^a \in \mathfrak{g}^*$, transform as dual Lie algebra basis elements.

These definitions ensure that the differentials of the bicomplex commute: $\mathrm{d}_G\,\delta = \delta\,\mathrm{d}_G$, so that the following diagram commutes. This is a bigraded complex (the l CE ghost-degree index increases upwards and the m Koszul ghost momenta-degree index increases leftwards from the bottom right corner) :

$$\begin{array}{ccccccc}
\Lambda^K(\mathfrak{g})\Lambda^K(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} & \Lambda^{K-1}(\mathfrak{g})\Lambda^K(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} \cdots \xrightarrow{\delta} & \Lambda^0(\mathfrak{g})\Lambda^K(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} & 0 \\
\mathrm{d}_G\Big\uparrow & & & \mathrm{d}_G\Big\uparrow & & & \\
\Lambda^K(\mathfrak{g})\Lambda^{K-1}(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} & \Lambda^{K-1}(\mathfrak{g})\Lambda^{K-1}(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} \cdots \xrightarrow{\delta} & \Lambda^0(\mathfrak{g})\Lambda^{K-1}(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} & 0 \\
\mathrm{d}_G\Big\uparrow & & & \mathrm{d}_G\Big\uparrow & & & \\
\Lambda^K(\mathfrak{g})\Lambda^{K-2}(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} & \Lambda^{K-1}(\mathfrak{g})\Lambda^{K-2}(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} \cdots \xrightarrow{\delta} & \Lambda^0(\mathfrak{g})\Lambda^{K-2}(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} & 0 \\
\mathrm{d}_G\Big\uparrow & & & \mathrm{d}_G\Big\uparrow & & & \\
\cdots\cdots & & & \cdots\cdots & & & \\
\mathrm{d}_G\Big\uparrow & & & \mathrm{d}_G\Big\uparrow & & & \\
\Lambda^K(\mathfrak{g})\Lambda^0(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} & \Lambda^{K-1}(\mathfrak{g})\Lambda^0(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} \cdots \xrightarrow{\delta} & \Lambda^0(\mathfrak{g})\Lambda^0(\mathfrak{g}^*)C^\infty(\mathbb{M}) & \xrightarrow{\delta} & 0
\end{array}$$

The horizontal sequences are the Koszul sequences and the vertical are the Chevalley-Eilenberg sequences.

4.2.5 The BRST complex

The BRST complex is the bicomplex above, with a different differential constructed from the two differentials of the bicomplex.

The BRST sequence indexed by $-K, \ldots, 0, \ldots, K$, is in the diagonal direction, in the diagram above, from bottom left to top right. The degree k BRST subspace of the BRST complex is the direct sum, $\bigoplus_{l-m=k} \Lambda^m(\mathfrak{g}) \otimes \Lambda^l(\mathfrak{g}^*) \otimes C^\infty(\mathbb{M})$, of terms above on the same diagonal for which $k = l - m$.

We wish to construct the differential δ_B over the bicomplex. We require that $\delta_B^2 = 0$ and that the zeroth cohomology is $C^\infty(\mathbb{M}_{TG})$. This can be achieved if $\delta_B = \mathrm{d}_G \ + \ (-1)^l\,\delta$, so long as both derivatives are nilpotent, $\mathrm{d}_G^2 \ = 0 = \ \delta^2$, and commute, $\mathrm{d}_G\,\delta = \ \delta\,\mathrm{d}_G$. The BRST grading

k is the difference between the C-E ghost degree l and the Koszul ghost-momenta degree m: $k = l - m$, the number l of C-E generators c^a minus the number m of Koszul generators ρ_a which are factors in a term. This is called the total ghost number. This complex is called the BRST complex and δ_B is the BRST differential. The BRST cohomology is

$$\begin{aligned} H^k_{BRST} &\cong 0 && k < 0 \\ &\cong H^k_{CE}(\mathfrak{g}, C^\infty(\mathbb{M}_T)) && k \geq 0 \end{aligned} \tag{4.16}$$

and so $H^0_{BRST} \cong H^0_{CE}(\mathfrak{g}, C^\infty(\mathbb{M}_T)) \cong C^\infty(\mathbb{M}_{TG})$, the ring of observables on physical phase space, as desired. The BRST cohomology is equal to the Lie algebra cohomology because the homology rings of the Koszul complex are zero except for the zeroth: $H_0(CE) = C^\infty(\mathbb{M}_T)$.

A remaining issue is whether the requirement $\mathrm{d}_G^2 = 0$ holds. If the symplectomorphism basis fields satisfy the Jacobi identity $[[X_a, X_b], X_c] + [[X_b, X_c], X_a] + [[X_c, X_a], X_b] = 0$, as when for instance the structure functions $f^a_{bc}(x)$ are constants, then $\mathrm{d}_G^2 = 0$ and so the bicomplex descends to homology. This means that the zeroth cohomology ring of the C-E complex is still $C^\infty(\mathbb{M}_{TG})$ even though the ring $C^\infty(\mathbb{M}_T)$ which was originally used to define the C-E complex is now only present as the zeroth homology ring of the Koszul complex. In general the Jacobi identity is not satisfied and so $((-1)^l\delta + \mathrm{d}_G)^2 \neq 0$. To obtain the necessary property, $\delta_B^2 = 0$, required for the BRST method to work, we add extra derivations to $(-1)^l\delta + \mathrm{d}_G$ to obtain $\delta_B = \delta + \mathrm{d}_G + \mathrm{d}_2 + \mathrm{d}_3 + \ldots$ until $\delta_B^2 = 0$. The extra derivations encode the higher relations of the algebra of the symplectomorphism basis fields and is described in detail in [20].

Poisson structure

In the above the construction of the BRST complex, the Poisson structure on $C^\infty(\mathbb{M})$ can be extended naturally to a graded Poisson structure on the bicomplex, with a graded Poisson bracket. The Poisson bracket on the odd generators is $\{c^a, \rho_b\} = \langle c^a, \rho_b\rangle = \delta^a_b$. The differential δ_B is an inner derivation, which means it can be implemented via the graded Poisson bracket: $\delta_B = -\{Q, \cdot\}$. Q, called the classical BRST charge, is a generator with total ghost number $k = +1$. Q needs to be nilpotent: $2Q^2 = \{Q, Q\} = 0$, because $0 = \delta_B^2 = -\{Q, -\{Q, \cdot\}\} = \{\{Q, Q\}, \cdot\} - \{Q, \{Q, \cdot\}\} = \frac{1}{2}\{\{Q, Q\}, \cdot\}$. Because the BRST differential is a Poisson derivation, the kernels $Z^k = \mathrm{Ker}\ \delta^k_B$ are graded Poisson algebras, and $B^k = \mathrm{Im}\ \delta^{k+1}_B$ are Poisson ideals, consequently the graded Poisson structure descends to the cohomology rings $H^k_{\delta_B} = Z^k/B^k$. In particular the ring of physical observables has a Poisson structure. In the case when the structure functions are constant, the BRST generator is $Q = c^aT_a - \frac{1}{2}f^a_{bc}c^bc^c\rho_a$. Using the differentials defined in previous sections, the Leibniz rule and the Jacobi identity, we obtain, as required,

$$\delta_B Q \ = \ (d_G + (-1)^l\delta)Q = \ (d_G + (-1)^l\delta)(c^aT_a - \frac{1}{2}f^a_{bc}c^bc^c\rho_a) = 0 \tag{4.17}$$

4.2.6 Quantization

Canonical quantization is performed by finding representations on a Krein space (which is a generalization of Hilbert spaces to spaces which are non-positive-definite) of the Lie algebra of the graded Poisson algebra of functions on super-phase space $(CE_{KT}, \{\})$. The graded Poisson bracket is promoted to a graded commutator: $\{\cdot,\cdot\} \longrightarrow \frac{1}{i\hbar}[\cdot,\cdot]$. The observables are promoted to operators acting on a Krein space. There can be zero and negative norm states so technically it is a generalization of a Hilbert space, a Krein space, rather than a Hilbert space that is required. The canonical quantization preserves the super Lie structure of the bicomplex CE_{KT}, a works so long as the original phase space can be canonically quantized. The Homology structure is preserved and the BRST function Q is promoted to an operator which has the same purpose: to project out the physical states. Usually the negative norm states will be projected out by Q for a physical theory, and the gauge variation will add zero norm states and will therefore not be physically significant (ie contribute to measurable amplitudes) although they may be useful in the formalism. This is the cohomological form of the Kugo-Ojuma treatment to obtain a correct inner product for states [49].

4.3 BRST example

These are presented in some detail so that the comparison with the analogous multisymplectic examples in sections 3.8 and 4.7 can be made.

(Note: we use the flat metric convention $g = [1, -1, -1, -1]$).

4.3.1 The electromagnetic field

This is from a detailed explanation of BRST for the electromagnetic field is in [70].

Configuration space action

The configuration space action is

$$S[A_\mu(x)] = \int_{M^4} \mathcal{L}(A_\mu(x))\mathrm{d}^d x = \int_{M^4} |\mathrm{d}A|^2 \mathrm{d}^d x = \int_{M^4} \partial_{[\mu}A_{\nu]}\, \partial_{[\lambda}A_{\rho]}\, g^{\mu\lambda} g^{\nu\rho}\, \mathrm{d}^d x = \int_{M^4} \frac{1}{2}(E_i E^i - B_i B^i)\mathrm{d}^d x, \tag{4.18}$$

where A is a 1-form field on $d = 4$ dimensional Minkowski spacetime M^4, $E_i := \partial_0 A_i - \partial_i A_0$, and $B^i := 2\epsilon^{ijk}\partial_j A_k$.

The Lagrangian density $\mathcal{L}$ is invariant under a gauge variation $\delta A = \mathrm{d}f = \partial_\mu f(x)\mathrm{d}x^\mu$ where $f \in C^\infty(M^4)$ is an arbitrary smooth function on spacetime.

Phase space

The phase space is $\mathbb{M} = \{(A_{\mu x}, p^{\mu x})\} = \{(A_{\mu x}, E^{ix}, p^{0x})\}$ where x 'indexes' the spatial points in Minkowski space. The time derivative of A_{0x} does not appear in the configuration space Lagrangian and so there is a primary constraint on performing the Legendre transformation: $p^{0x} \approx \frac{\partial \mathcal{L}}{\partial A_{0x,0}} = 0$.

The other canonical momenta are $p^{ix} \approx \frac{\partial \mathcal{L}}{\partial A_{ix,0}} = (\partial_\nu A_{\mu x} - \partial_\mu A_{\nu x})\ g^{\nu i} g^{\mu 0} =: E^{ix}$ which is $E^{ix} = -(\partial_i A_{0x} - \partial_0 A_{ix})$ in Minkowski spacetime with metric $g =$ diag$[1,-1,-1,-1]$.

We introduce a change of notation: we now write integration over a flat spatial slice $M^{d-1} = \mathbb{R}^{d-1} \subset M^d$ as an infinite summation over an 'index' x, the coordinate of each spatial point, employing the summation convention analogously to the index i. The spatial volume form $\mathrm{d}^{d-1}x$ should be considered as incorporated, as a factor, into one of the factors in each term.

The Hamiltonian is

$$H = \int_{M^3} (p^\mu \partial_0 A_\mu \ - \ \mathcal{L}(A_\mu(x))\mathrm{d}^{d-1}x = \frac{1}{2}(E^{ix}E_{ix} + B^{ix}B_{ix}) + \partial_i A_{0x} E^{ix} + p^{0x}\partial_0 A_{0x}$$
$$= \frac{1}{2}(E^{ix}E_{ix} + B^{ix}B_{ix}) - A_{0x}\partial_i E^{ix} - \partial_0 p^{0x} A_{0x}, \tag{4.19}$$

employing integration by parts over spacetime inside the first order action (4.24) for the last equality. E^{ix} is now viewed as the momenta conjugate to A_{ix} and is another notation for p^{ix}, and p^{0x} is the momentum conjugate to A_{0x}, whereas B^{ix} is conventional shorthand: $B^{ix} := 2\epsilon^{ijk}\partial_j A_{kx}$. The term with time derivatives is usual in the case of a primary constraint, because not all the velocities can be replaced by momenta, from the non-invertibility of the Legendre transformation in that case. The gauge transformations on phase space are

$$\delta_f A = \mathrm{d}f = \partial_\mu f(x)\mathrm{d}x^\mu \quad , \quad \delta_f p^\mu = 0 \quad , \quad \delta_f x^\mu = 0 \tag{4.20}$$

where $f(x)$ is an arbitrary smooth function on spacetime.

The transformations of the momenta are chosen for on-shell compatibility, because $\delta_f p^{ix} \approx \delta_f(-\partial_0 A_{ix} + \partial_i A_{0x}) \ = \ -\partial_0\partial_i f(x) + \partial_i \partial_0 f(x)) \ = \ 0$. We also have $\delta_f E^{ix} = \delta_f[\ -(\partial_i A_{0x} - \partial_0 A_{ix}\ ] = 0 = \delta_f B^{ix} := \delta_f[\ 2\epsilon^{ijk}\partial_j A_{kx}\ ]$. So E^{ix} and B^{ix} are gauge invariant observables.

The corresponding variation of the Hamiltonian is

$$\delta_f H = -\ \partial_{0x} f\ (\partial_i E^{ix} + \partial_0 p^{0x}) \tag{4.21}$$

This Hamiltonian can be seen to be invariant under gauge transformations which are constant in time, $\partial_0 f = 0$. It can be seen that a secondary constraint (Gauss's law), $\partial_i E^{ix} = 0$, is required, together with the primary constraint $p^{0x} = 0$, to make the Hamiltonian gauge invariant for time varying gauge variations. When the primary and secondary constraints are satisfied, the above Hamiltonian is gauge invariant: $\delta H = 0$ on the constraint surface in phase space: $\partial_i E^{ix} = 0$, $p^0 = 0$. In terms of Poisson brackets, and the the gauge variation of an observable O, at the point x in spacetime, can be written as

$$\delta_f O = \delta_f A_{\mu x} \frac{\partial O}{\partial A_{\mu x}} = (\partial_\mu f(x)) \frac{\partial O}{\partial A_{\mu x}} = -\{\ (\partial_\mu f(x))\ p^{\mu x}\ ,\ O\ \} = \{f(x)(\partial_i E^{ix} + \partial_0 p^{0x}), O\} \tag{4.22}$$

It can be seen that the constraints $T_x := \partial_i E^{ix} + \partial_0 p^{0x}$ generate the gauge variation. (Note that here, in spite of the apparent notation, $p^{\mu x}$ are ordinary momenta canonically dual to $A_{\mu x}$ and are not multimomenta.) The Poisson bracket here is

$$\{\ A\ ,\ B\ \} := A \cdot \frac{\overleftarrow{\partial}}{\partial A_{\mu x}} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\mu x}} \cdot B \tag{4.23}$$

(integration over spatial points x is assumed from the notation above with the summation convention).

First order Lagrangian

The corresponding first order Lagrangian is

$$L_P = \int_{M^3} (p^\mu \partial_0 A_\mu \ - \mathcal{H}) \mathrm{d}^{d-1}x = p^{\mu x} \partial_0 A_{\mu x} \ - \frac{1}{2}(E^{ix} E_{ix} + B^{ix} B_{ix}) - \partial_i A_{0x} E^{ix} - p^{0x} \partial_0 A_{0x}$$
$$= E^{ix} \partial_0 A_{ix} \ - \frac{1}{2}(E^{ix} E_{ix} + B^{ix} B_{ix}) - \partial_i A_{0x} E^{ix} = E^{ix} \partial_0 A_{ix} \ - \frac{1}{2}(E^{ix} E_{ix} + B^{ix} B_{ix}) + A_{0x} \partial_i E^{ix} \tag{4.24}$$

The first order Lagrangian is invariant under general (i.e. time dependent) gauge transformations, $\delta_f L_P = 0$. Note that the first order Lagrangian does not contain the (primary constraint) momentum, p^{0x}, canonically conjugate to A_{0x}. As a result A_{0x} only serves as a Lagrange multiplier to enforce the secondary constraint.

BRST super-phase space

We now enlarge the phase space to a BRST super-phase space by adjoining extra odd canonical pairs, c_x, ρ^x and $\bar{c}_x, \bar{\rho}^x$ so that the new super-phase space has coordinates $A_{\mu x}, p^{\mu x}, c_x, \rho^x, \bar{c}_x, \bar{\rho}^x$. The odd coordinates are the generators of the BRST bicomplex.

The global BRST variation δ_B on the super-phase space is defined to be

$$\delta_B A_x = (\mathrm{d}c)_x = (\partial_\mu c)_x\ \mathrm{d}x^\mu \tag{4.25}$$

$$\delta_B p^{ix} = 0 \tag{4.26}$$

$$\delta_B c_x = 0 \tag{4.27}$$

$$\delta_B \bar{c}^x = -p^{0x} \tag{4.28}$$

$$\delta_B p^{0x} = 0 \tag{4.29}$$

$$\delta_B \rho^x = 0 \tag{4.30}$$

$$\delta_B \bar{\rho}^x = 0 \tag{4.31}$$

Here the gauge variation parameter $f(x)$ is replaced by $c(x)$, which has opposite grassmann parity to $f(x)$, in the specification of the global BRST variation. This ensures that the phase space Lagrangian above is invariant under both the gauge variation and also, viewed as a function on the super-phase space, the global BRST variation.

Gauge fixing

We want to remove the gauge freedom $\partial_\mu f(x)$ of the field A_μ. There are various possible choices, for example such as choosing 'temporal gauge' $A_0 = 0$. Here we will choose $\partial^\mu A_\mu = 0$, which is called Lorenz gauge or transverse gauge and has the advantage that it is covariant and that it is the same as $k^\mu \tilde{A}_\mu = 0$ in frequency space (these and other properties lead to simpler calculations in QFT). This could be done using a Lagrange multiplier p^x and adding the term $p^x \partial^\mu A_{\mu x}$ to the Lagrangian so that the equation of motion for p^x is then $\partial^\mu A_\mu = 0$. However is advantageous to use 'R_ξ' gauge fixing by adding a term $\frac{\xi}{2} p^{0x} p_{0x} - p^{0x} \partial^\mu A_{\mu x}$ to the Lagrangian instead, where ξ is a fixed parameter. Strictly speaking, this is gauge breaking rather than gauge fixing, but the phrase 'gauge fixing' is still used. This form of gauge fixing has a desirable gaussian form for calculating path integrals in QFT. The gauge degrees of freedom are removed in the sense that the Lagrangian is no longer symmetric in those degrees of freedom, which are still present. These now appear as 'physical' degrees of freedom in the internal lines of Feynman diagrams, but are not present in the external lines, the asymptotic states. The removal of the symmetry (the degeneracy of the Lagrangian) allows the Lagrangian to be inverted so that the propagator can be obtained, which is then used in the Feynman expansion.

We can add the following BRST-exact term to the phase-space Lagrangian to effect gauge fixing to Lorenz gauge, $(\partial^0 A_{0x} + \partial^i A_{ix} \approx 0)$:

$$\delta\Psi = \delta_B\{\bar{c}^x(\partial^0 A_{0x} + \partial^i A_{ix} - \frac{\xi}{2} p^{0x})\} = \frac{\xi}{2} p^{0x} p_{0x} - p^{0x}(\partial^0 A_{0x} + \partial^i A_{ix}) - \bar{c}^x(\partial^\mu \partial_\mu c_x) \tag{4.32}$$

ξ is a fixed parameter.

The BRST invariant gauge fixed Lagrangian is :

$$L_B = \int_{M^3} (\mathcal{L}_P - \delta\Psi)\ \mathrm{d}^{d-1}x =$$

$$p^{ix}\partial_0 A_{ix}\ -\frac{1}{2}(p^{ix}p_{ix}+B^{ix}B_{ix})+A_{0x}\partial_i p^{ix}-\frac{\xi}{2}p^{0x}p_{0x}+p^{0x}(\partial^0 A_{0x}+\partial^i A_{ix})+\partial^\mu \bar{c}^x\partial_\mu c_x \quad (4.33)$$

Inserting the ghost momenta in order to have the ghost part to first order in the time derivatives in the Lagrangian:

$$L_{BP} = \rho_c^x\partial_0 c_x + \bar{\rho}_x\partial_0\bar{c}^x + p^{0x}\partial_0 A_{0x}\ + p^{ix}\partial_0 A_{ix}$$

$$-\frac{1}{2}(p^{ix}p_{ix}+B^{ix}B_{ix}) - \partial_i A_{0x}p^{ix} - \frac{\xi}{2}p^{0x}p_{0x} + p^{0x}\partial^i A_{ix} - \rho_c^x\bar{\rho}_x + \partial^i\bar{c}^x\partial_i c_x \quad (4.34)$$

From this we read off the BRST gauge fixed Hamiltonian:

$$H_B = \frac{1}{2}(E^{ix}E_{ix}+B^{ix}B_{ix}) - A_{0x}\partial_i E^{ix} + \frac{\xi}{2}p^{0x}p_{0x} + p^{0x}\partial^i A_{ix} + \rho^x\bar{\rho}_x - \partial^i\bar{c}^x\partial_i c_x \quad (4.35)$$

where we used the notation E^{ix} for p^{ix}.

By using the Euler-Lagrange equations for the momenta $p^{ix}, p^{0x}, \rho^x, \bar{\rho}^x$, in (4.34) and substituting back into (4.34) to eliminate these, we obtain the BRST configuration space Lagrangian:

$$L_{BC} = \int_{M^3} (\mathcal{L} - \delta\Psi)\ \mathrm{d}^{d-1}x =$$

$$\frac{1}{2}(E^{ix}E_{ix}\ -\ B^{ix}B_{ix})\ +\ \frac{1}{2\xi}\partial^\mu A_{\mu x}\partial_\nu A^{\nu x}\ +\ \partial^\mu\bar{c}^x\partial_\mu c_x$$

$$= 2\partial^{[0}A^{i]x}\partial_{[0}A_{i]x}\ -\ 2\epsilon^{ijk}\partial_j A_{kx}g_{ii'}\epsilon^{i'j'k'}\partial_{j'}A_{k'x}\ +\ \frac{1}{2\xi}(\,\partial^\mu A_{\mu x})^2\ +\ \partial^\mu\bar{c}^x\partial_\mu c_x$$

$$= \partial^{[\mu}A^{\nu]x}\partial_{[\mu}A_{\nu]x}\ +\ \frac{1}{2\xi}(\,\partial^\mu A_{\mu x})^2\ +\ \partial^\mu\bar{c}^x\partial_\mu c_x \quad (4.36)$$

$$= \frac{1}{4}F^{\mu\nu\,x}F_{\mu\nu\,x}\ +\ \frac{1}{2\xi}(\,\partial^\mu A_{\mu x})^2\ +\ \partial^\mu\bar{c}^x\partial_\mu c_x \quad (4.37)$$

In this case the odd variables are decoupled from the other variables and could be ignored. Then it can be seen that the end result is the addition of a term to the velocity phase space Lagragian, where the classical minimization of the action forces the average of $|\partial^\mu A_{\mu x}|$ to be small. For QFT, in the quantum functional integral, this becomes a gaussian factor $\exp\{\ \frac{-i}{\hbar}\frac{1}{2\xi}(\,\partial^\mu A_{\mu x})^2\}$ in the integrand, which, when integrated out, becomes a delta function $\delta(\partial^\mu A_{\mu x})$ in the integrand, which enforces the constraint $\partial^\mu A_{\mu x} = 0$ on the path integral (this was first proposed by DeWitt [13] [14] [15]). An overall constant will appear which can be ignored in the abelian Yang-Mills case, because it does not change if we change the gauge fixing function $C(x) = \partial^\mu A_\mu(x)$. In the case that the gauge fixing function does vary with a gauge transformation then there is a Jacobian factor $J = \det\left(\frac{\partial C^a_x}{\partial f^b}\right)$ in the integrand of the path integral, where $a = 1\ldots K$ label the gauge fixing functions C^a, and the gauge variation parameters f^a. This factor can itself be written as gaussian integral with grassman odd variables $c^a, \bar{c}^a$:

$$J = \det\left(\frac{\partial C^a}{\partial f^b}\right) = \int \mathrm{d}c^1\ldots\mathrm{d}c^K\mathrm{d}\bar{c}_1\ldots\mathrm{d}\bar{c}_K\exp\{\bar{c}_a\frac{\partial C^a}{\partial f^b}c^b\} \quad (4.38)$$

In the functional integral this is a 'Fadeev-Popov' [64] term $\bar{c}_a \frac{\partial C^a}{\partial f^b} c^b$ in the Lagrangian density and it is this term that the BRST gauge fixing produces automatically.

If we set $\xi = \frac{1}{2}$ (Feynman gauge) then, using partial integration, we will have $\mathcal{L}_{BC} = \frac{1}{2}(\partial_\mu A_\nu)(\partial^\mu A^{\nu x})$ (ignoring the ghost term which decouples) which we will also later obtain using the multiphase-space gauge fixing in section 4.7.1, eq. (4.111).

In the canonical quantization the odd observables here lead to negative norm states, which have to be mod'ed out. This is achieved in the quantum BRST cohomology [49].

The Poisson generator of the BRST variation is:

$$Q = -i(c_x \partial_i E^{ix} + \bar{\rho}_x p^{0x}) \tag{4.39}$$

which commutes with H_B: $\delta_B H_B = -i\{Q, H_B\} = 0$. In this particular case (abelian Yang-Mills), the conjugate $\bar{Q} = i(\bar{c}^x \partial_i E^{ix} + \rho^x p^{0x})$ also commutes with H_B: $\bar{\delta}_B H_B = -i\{\bar{Q}, H_B\} = 0$.

The Poisson bracket here is on the super-phase space, with extra terms $\frac{\partial}{\partial c_x} \wedge \frac{\partial}{\partial P_c^x} + \frac{\partial}{\partial \bar{c}^x} \wedge \frac{\partial}{\partial P_{\bar{c}x}}$.

4.4 Multiphase-space BRST

This section generalizes the well known phase-space BRST formalism for dynamical systems with gauge symmetries described above in the previous sections, to a multiphase-space BRST formalism on an extended supermultiphase space. Some particular examples are presented: abelian and non-abelian Yang-Mills.

The introduction to BRST in section (4.2) above is presented as if the symplectic manifolds involved are finite dimensional, whereas the main application of BRST is to analyze local field theories, which can be viewed as Hamiltonian systems with an infinite number of degrees of freedom. As shown in the example of conventional BRST applied to the electromagnetic field in appendix D.2, in local field theories there are separate configuration and momentum variables at each spatial point, and often local gauge variations which can be made semi-independently at each spatial or spacetime point. For example, in Yang-Mills theory described in section 3.8, the gauge variation is $\delta A(x) = -Df(x)$, where $A(x)$ is the Lie algebra valued spacetime-1-form field, which is a connection on a principal vector bundle over spacetime, D is the covariant derivative, and $f(x)$ is any smooth Lie algebra valued function on spacetime. The gauge group parametrized by $f(x) = f^a(x)\bar{e}_a$, here is the group of smooth vertical automorphisms of the principal bundle, and is infinite dimensional. The gauge group restricted to a fiber is usually finite dimensional and is called the structure group. The $\bar{e}_a$ are a basis of the Lie algebra of the structure group. In BRST, for example that of the electromagnetic field D.2, the number of ghosts is infinite and parametrized by the vector index of the Lie algebra and the spatial points: $c^a(x)$, and the BRST charge is usually an integral over a spatial slice of some function of the of field and ghost canonical pairs and their spatial partial derivatives at each point. The Poisson brackets exist at each moment in time and are also integrated over spatial slices M^{d-1} of Minkovski spacetime:

$$\{ O_1 \ , \ O_2 \} \ = \ O_1 \ \cdot \frac{\overleftarrow{\partial}}{\partial A^a_{\mu x}} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\mu x}_a} \cdot \ O_2 \ := \int_{M^{d-1}} \ O_1 \ \cdot \frac{\overleftarrow{\partial}}{\partial A^a_{\mu}} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\mu}_a} \cdot \ O_2 \ \mathrm{d}^{d-1}x \tag{4.40}$$

The multiphase-space version aims to have a BRST-type theory where the objects are not spatial integrals but rather exist at each spacetime point, i.e. look at functions on the fiber rather than spatial sections of the entire bundle. However the field at neighbouring spacetime points (on neighbouring fibers) is relevant and this is incorporated by the use of spacetime derivatives and indices, and, in particular, multimomenta. The multiphase-space BRST is on a bundle over spacetime rather than a phase space over a one dimensional (time) base space, and we would expect a BRST generator as a function on multiphase space and a multibracket defined at each spacetime point.

The generalization of BRST which we consider here is to go, from a BRST complex over the Poisson algebra of functions on phase space, to a BRST complex over a bracket algebra of 'functions on multiphase space', where the latter is some suitably chosen algebra so that the BRST construction can work. One could consider a range of possibilities with two extremes: a restrictive version 'A' where some algebra R of multiphase-space observables is chosen so that the BRST construction is the same or very similar to the conventional BRST, except that it is on a bundle over spacetime as mentioned in the previous paragraph. Or a more general version 'B' where the algebra $R = C^\infty(\mathbb{M})$ of functions on multiphase space $\mathbb{M}$ is chosen and the BRST construction is modified. In this thesis we consider possibility 'A'.

Most of the BRST construction generalizes readily and it is useful to examine in detail the particular parts of this construction where the generalization is not straightforward, and summarize those aspects which are the same without repeating the detail of the construction which is already present in section 4.2.

4.4.1 Multiphase-space BRST construction

Multiphase-space BRST follows closely the description of conventional BRST in the previous sections. The difference is primarily in the Koszul complex. The ghost part is the same as in the conventional BRST, but the because the 'body' part (with no grassmann odd factors) is different, in particular in that observables and constraints are spacetime $d-1$-forms, it is necessary to define in particular the Koszul complex to ensure that the complexes have the requisite structure for the BRST zeroth cohomology ring to represent the physical observables.

Koszul complex

The Koszul complex is the free R-module generated by forms ρ_a's , $K_T := R\langle\rho_a\rangle$ for $\rho_a,\ a = 1\ldots K$ and where multiplication of the ρ_a is the exterior product, and where R is the ring $\Lambda_B^{d-1}(\mathbb{M})$ of spacetime $d-1$ -forms whose coefficients are functions on multiphase space: locally of the form $f^\mu(x^\nu, u^i, p_i^\nu)\mathrm{d}^{d-1}x_\mu$. The multiplication in this ring is the wedge product of $d-1$ -forms and is trivial.

The $\rho_a,\ a = 1\ldots K$ are particular spacetime $d-1$ -forms $\rho_a = \rho_a^\mu \mathrm{d}^{d-1}x_\mu$ on the super-multiphase space, in local Darboux coordinates, $\{(x^\nu, u^i, p_i^\nu, c^a, \rho_a^\mu)\}$. Because multiplication is the exterior product, it is trivial here: $\rho_a\rho_b = \rho_a \wedge \rho_b = 0$ (this is because the top spacetime form is degree d).

There is a vertical bracket defined on the fibers over spacetime by $\{u^i, p_j^\mu \mathrm{d}^{d-1}x_\mu\} = \delta_j^i$, and $\{u^i, p_j^\mu\}_\nu = \delta_j^i\delta_\nu^\mu$. The graded brackets for the odd coordinates on super-multiphase space: $\{c^a, \rho_b^\mu \mathrm{d}^{d-1}x_\mu\} = \delta_b^a$, and $\{c^a, \rho_b^\mu\}_\nu = \delta_b^a\delta_\nu^\mu$. The bracket is extended to functions on multiphase space by the Leibnitz rule on products.

The constraints are a set of K spacetime $(d-1)$ -forms on multiphase space,

$$\{T_a^\mu(x^\nu, u^i, p_i^\mu)\mathrm{d}^{d-1}x_\mu, a = 1, \ldots, K\} \tag{4.41}$$

corresponding to the Lie algebra $\mathfrak{g}$ of infinitesimal hamiltonian multisymplectomorphisms represented by a basis of vector fields X_a on multiphase space. We here limit to the case where the functions $T_a^\mu(x^\nu, u^i, p_i^\mu)$ are linear in the multimomenta p_i^μ, so that the general form of T_a is $T_a = T_a^\mu(x^\nu, u^i, p_i^\mu)\mathrm{d}^{d-1}x_\mu = \Pi_{a\alpha}^{\mu j}(x^\nu, u^i)\ p_j^\alpha\ \mathrm{d}^{d-1}x_\mu$. The T_a's are also required to satisfy the conditions of a set of first class constraints, $\{T_a, T_b\} = -f_{ab}^c T_c$, where f_{ab}^c are the structure constants for a Lie algebra $\mathfrak{g}$ and that they generate infinitesimal hamiltonian multisymplectomorphisms via $X_a\cdot = -\{T_a, \cdot\}$. A BRST system with both primary and secondary constraints is developed in subsection 4.5.1.

Because of the restriction on the form of T_a imposed in the previous paragraph, that it be linear in the multimomenta p_i^μ, we obtain

$$\begin{aligned}
\delta_a x^\mu &= -\{T_a, x^\mu\} = 0 \\
\delta_a u^i &= -\{T_a, u^i\} = \Pi_{a\beta}^{\beta i}
\end{aligned}$$

$$\begin{aligned}
\delta_a p_i^\alpha &= -\{T_a, p_i^\alpha\} = -\Pi^{\alpha j}_{a\beta,i} p_j^\beta \\
\delta_a T_b &= -\{T_a, T_b\} = -2\Pi^{\gamma j}_{[a|\beta,k} p_j^\beta \Pi^{\alpha k}_{|b]\gamma}\ \mathrm{d}x_\alpha
\end{aligned} \tag{4.42}$$

$\{T_a, T_b\} = -f^c_{ab} T_c$ implies that the functions Π have to satisfy

$$2\Pi^{\gamma j}_{[a|\alpha,k} \Pi^{\mu k}_{|b]\gamma} = -f^c_{ab} \Pi^{\mu j}_{c\alpha} \tag{4.43}$$

The action of $\delta_a \cdot = X_a \cdot = -\{\cdot, T_a\}$ on multiphase space defines the action on R via the action given above on the coordinates (x^ν, u^i, p_i^μ) together with the Leibnitz rule. The X_a are the basis for a Lie algebra of infinitesimal hamiltonian multisymplectomorphisms where $[X_a, X_b] = f^c_{ab} X_c$, where f^c_{ab} are the structure constants of the Lie algebra in the basis $\{X_a\}$.

The space of spacetime $d-1$ -forms on the constraint surface $T_a = T_a^\mu(x^\nu, u^i, p_i^\nu)\ \mathrm{d}^{d-1}x_\mu = 0, a = 1 \ldots K$ is isomorphic to the space of equivalence classes $R_T \cong R/I$ where $I = R\langle T\rangle$ is the ideal generated by the set of constraints $T = \{T_a^\mu \mathrm{d}^{d-1}x_\mu \mid a = 1\ldots K\}$. $R_T \cong R/I$ will become the zeroth homology ring of the Koszul complex in exactly the same way as in the conventional BRST.

The differential δ on the Koszul complex is defined as $\delta f = 0$, for $f \in R$ and $\delta\rho_a = T_a = T_a^\mu \mathrm{d}^{d-1}x_\mu$, where T_a are the spacetime $d-1$ -form constraints, extended to K_T as an odd graded $\mathbb{R}$-linear derivation, which forces the differential to be nilpotent: $\delta\delta = 0$ and reduces the ρ -degree l by one. This is the same as for the conventional BRST above and the whole BRST construction above carries through on the space $R = \Lambda_B^{d-1}(\mathbb{M})$ as it did for the space of functions $R = \mathbb{R}\langle q^i, p_i\rangle$ on phase space as described for conventional BRST in section 4.2. (In the multisymplectic case, because ρ_a are $d-1$ -forms, when $d > 1$ the product of the Koszul generators is trivial: $\rho_a\rho_b = \rho_a \wedge \rho_b = 0$.)

The construction of the complexes is exactly as in section 4.2 and the only aspect that needs clarification is the notion of a multiplicative ideal I of functions which are zero on the constraint surface defined by $T_a = 0,\ a = 1, \ldots, K$ (shortened to $T = 0$) and that the equivalence classes R/I are to have the requisite properties.

First considering I, we want the Lie algebra action to preserve I and the constraint surface. To show this, we use the brackets for the constraint $d-1$-forms T_a's, the hamiltonian property of the T_a's and the Leibnitz property of the brackets.

Let $t = f^a T_a \in I$ and $f^a, g^a \in C^\infty(\mathbb{M})$, then the variation of a 'function' (actually a $d-1$ -form) in the ideal I is $\delta_b t = \delta_b(f^a T_a) = (\delta_b f^a) T_a + f^a(\delta_b T_a) = (\delta_b f^a) T_a + f^a\{T_a, T_b\} = (\delta_b f^a) T_a + f^a f^c_{ab} T_c\} \stackrel{T=0}{=} 0$, which is zero on the constraint surface $T = 0$ as indicated by the last equality symbol. Also clearly $\delta_b t \in I$ so I is preserved by the action (the action is compatible with I) and so the orbits of the group action are either entirely in the constraint surface $T = 0$ or disjoint from it. In addition, the bracket of two functions in I is $\{t_1, t_2\} = \{f_1^a T_a, f_2^b T_b\} = f_1^a\{T_a, T_b\}f_2^b + f_1^a\{T_a, f_2^b\}T_b + T_a\{f_1^a, T_b\}f_2^b + T_a\{f_1^a, f_2^b\}T_b = f_1^a f_{ab}^c T_c f_2^b + f_1^a\{T_a, f_2^b\}T_b + T_a\{f_1^a, T_b\}f_2^b + T_a\{f_1^a, f_2^b\}T_b \in I$, thus I is a sub bracket algebra of R. We cannot say that I is a bracket ideal because if $t = f^a T_a \in I$ and $g \in R$, $\{t, g\} = \{f^a T_a, g\} = f^a\{T_a, g\} + T_a\{f^a, g\} = -f^a X_a \cdot g + t_1$ for some $t_1 \in I$. Thus I is a bracket ideal if $X \cdot g \in I, \forall X \in \mathfrak{g}$ and $\forall g \in R$, which would imply that $X \cdot g \overset{T=0}{\equiv} 0$ which is not generally the case. If instead of R we restrict to the ring $N(I)$ of functions in R which are constant on the orbits of the group action in the constraint surface $T = 0$ then $X \cdot g \overset{T=0}{\equiv} 0, \forall X \in \mathfrak{g}$ and $\forall g \in N(I)$ and by the completeness of the T's, $X \cdot g \in I$.

Chevalley-Eilenberg complex

The Koszul complex has a Lie algebra $\mathfrak{g}$ action defined as in conventional BRST, (where the generators of the Koszul complex are viewed as a basis for the Lie algebra $\mathfrak{g}$,) by $\rho_a \cdot = \delta_a \cdot$ on R and by the adjoint action on ρ_b: $\rho_a \cdot \rho_b = [\rho_a, \rho_b] = f_{ab}^c \rho_c$, extended K_T by the graded Liebnitz rule (which becomes trivial for $d > 1$). We can therefore define the CE complex as in conventional BRST with ghosts $c^a \in \mathfrak{g}^*$.

BRST complex

The BRST complex is constructed as in the conventional BRST. The BRST variation $\delta = (-1)^l \delta + \mathrm{d}_G$ is constructed from the differentials, δ, d_G, on the bicomplex:

$$\begin{aligned}
\mathrm{d}_G u^i &= c^a \delta_a u^i \\
\mathrm{d}_G p_a^\mu &= c^a \delta_a p_a^\mu \\
\mathrm{d}_G T_b &= c^a f_{ab}^c T_c \\
\mathrm{d}_G c^a &= -\frac{1}{2}[c, c]^a = -\frac{1}{2} f_{bc}^a c^b c^c \\
\mathrm{d}_G \rho_a &= -\rho_b f_{ac}^b c^c \\
\mathrm{d}_G x^\mu &= 0 \\
\delta g &= 0 \text{ where } g \text{ is a function on multiphase space} \\
\delta \rho_a &= T_a \\
\delta c^a &= 0 \\
\delta x^\mu &= 0
\end{aligned} \tag{4.44}$$

The gauge algebra action on the bicomplex is defined in the same way as conventional BRST, and commutes with the differentials.

The BRST generator is $Q = c^a T_a - \frac{1}{2} f^c_{ab} \rho_c c^a c^b$, where $\{c^a, \rho_b\} = \{\rho_b, c^a\} = \delta^a_b$, which is the same as for conventional BRST, with of course the proviso that T_a and ρ_a are $d-1$ -forms and the bracket is a multiphase- space bracket acting on $d-1$ -forms. We also assume that f^c_{ab} is independent of $x^\mu, p^\mu_a, u^i, \rho_b, c^a$, otherwise there will be extra terms to the BRST generator is Q.

This is necessary to obtain $2Q^2 = \{Q, Q\} = 0$ require for the BRST construction, The definitions above ensure that $2Q^2 = \{Q, Q\} = 0$, in the same way as in the conventional BRST.

Because of the restriction on the form of T_a imposed above , that it be linear in the multimomenta p^μ_i, we obtain

$$\begin{aligned} -\delta_B u^i &= \{Q, u^i\} = c^a \{T_a, u^i\} = -c^a \Pi^{\beta i}_{a\beta} \\ -\delta_B p^\alpha_i &= \{Q, p^\alpha_i\} = c^a \{T_a, p^\alpha_i\} = c^a \Pi^{\alpha j}_{a\beta,i} p^\beta_j \end{aligned} \tag{4.45}$$

As before we have the BRST variations δ_B on the BRST super-multiphase-space coordinates:

$$\begin{aligned} -\delta_B c^c &= \{Q, c^c\} = -\frac{1}{2} f^c_{ab} c^a c^b \\ -\delta_B \rho_a &= \{Q, \rho_a\} = T_a + f^c_{ab} \rho_c c^b \\ -\delta_B T_b &= \{Q, T_b\} = c^a \{T_a, T_b\} = -c^a f^c_{ab} T_c \\ -\delta_B u^i &= \{Q, u^i\} = c^a \{T_a, u^i\} = -c^a \delta_a u^i \\ -\delta_B p^\alpha_i &= \{Q, p^\alpha_i\} = c^a \{T_a, p^\alpha_i\} = -c^a \delta_a p^\alpha_i \\ -\delta_B x^\alpha &= 0 \end{aligned} \tag{4.46}$$

From these we directly obtain, employing the graded Leibnitz rule for brackets

$$\begin{aligned} -\delta^2_B c^c &= 0 \\ -\delta^2_B \rho_a &= 0 \\ -\delta^2_B T_b &= 0 \\ -\delta^2_B x^\alpha &= 0 \end{aligned} \tag{4.47}$$

in the same way as in conventional BRST.

Similarly writing out the calculation for $-\delta^2_B u^i$ we obtain

$$\begin{aligned} -\delta^2_B u^i &= -c^a c^b \{T_b, \delta_a u^i\} - \frac{1}{2} f^c_{db} c^d c^b \delta_c u^i = c^a c^b \delta_b \delta_a u^i - \frac{1}{2} f^c_{db} c^d c^b \delta_c u^i \\ &= c^a c^b \delta_{[b} \delta_{a]} u^i - \frac{1}{2} f^c_{db} c^d c^b \delta_c u^i = c^a c^b \frac{1}{2} f^c_{ba} \delta_c u^i - \frac{1}{2} f^c_{db} c^d c^b \delta_c u^i = 0 \end{aligned} \tag{4.48}$$

In this calculation we used the Lie algebra property of the gauge variation, $\delta_{[b}\delta_{a]} := \frac{1}{2}(\delta_b\delta_a - \delta_b\delta_a) = \frac{1}{2}f^c_{ba}\delta_c$, and the hamiltonian property of the variation, $\{T_a, u^i\} = -\delta_a u^i$. A similar calculation for $-\delta_B^2 p_i^\alpha$ gives

$$-\delta_B^2 p_i^\alpha \;=\; 0. \tag{4.49}$$

Because $\delta_B^2 = 0$ for all coordinates on the BRST super-multiphase space, $\delta_B^2 = 0$ for any function of these coordinates. Because $\rho_a = \rho_a^\mu \mathrm{d}^{d-1}x_\mu$ and $T_a = T_a^\mu \mathrm{d}^{d-1}x_\mu$, we can write

$$-\delta_B \rho_a^\mu \;=\; \{Q, \rho_a^\mu\} = T_a^\mu + f^c_{ab}\rho_c^\mu c^b \tag{4.50}$$

There is also the multibracket for the odd coordinates of BRST super-multiphase space: $\{c^a, \rho_b^\mu\}_\nu = \{\rho_b^\mu, c^a\}_\nu = \delta_b^a \delta_\nu^\mu$.

Having defined the BRST variation on all the coordinates of super-multiphase space, we can extend this to functions of these BRST super-multiphase-space coordinates with the graded Leibnitz property of the brackets.

The BRST construction requires $\delta_B^2 \cdot = \{Q, \{Q, \cdot\}\} = \{\{Q, Q\}, \cdot\} - \{Q, \{Q, \cdot\}\} = 0$. But we have shown in (4.47) that δ_B^2 is zero for all the coordinate functions of BRST super-multiphase space. So we can claim the Jacobi identity $\{Q, \{Q, A\}\} = \{\{Q, Q\}, A\} - \{Q, \{Q, A\}\}$, for arbitrary functions, linear in the multimomenta, on BRST super-multiphase space.

The Jacobi identity for multiphase-space functions A, linear in the multimomenta, and variation generators T_a and T_b:

$$\{A, \{T_a, T_b\}\} + \{T_a\{T_b, A\}\} + \{T_b, \{A, T_a\}\} = -f^c_{ab}\{A, T_c\} + \delta_a\delta_b A - \delta_b\delta_a A$$

$$= -f^c_{ab}\delta_c A + \delta_a\delta_b A - \delta_b\delta_a A = -f^c_{ab}\delta_c A + [\delta_a, \delta_b]A = -f^c_{ab}\delta_c A + f^c_{ab}\delta_c A = 0 \tag{4.51}$$

Coisotropic Marsden-Weinstein reduction expressed in terms of Poisson algebras

Let I be the vanishing ideal of M_T, the ring of exact hamiltonian spacetime $(d-1)$ -forms on multiphase space M which are zero on M_T. (If $\langle T\rangle$ is the ideal generated by a set of regular constraints $T_a = 0$ which define M_T, then $I = \langle T\rangle$.) Then the ring of functions on the constraint surface is

$$R_T \cong R/I. \tag{4.52}$$

The coisotropy of M_T means that I is a Poisson sub algebra of $\mathbb{M}$, called a coisotropic ideal. The above is not a Poisson algebra because I is not a Poisson ideal of $\mathbb{M}$. However I is a Poisson ideal of the Poisson normalizer $N(I)$ of I in $\mathbb{M}$, where $N(I) := \{f \in C^\infty(\mathbb{M}) | \{f, I\} \subset I\}$, the Poisson subalgebra of $C^\infty(\mathbb{M})$ of functions which are constant on the orbits on $\mathbb{M}_T$ generated by $\{I, \cdot\}$ and therefore the quotient

$$C^\infty(\mathbb{M}_{TT}) \cong N(I)/I \tag{4.53}$$

is also a Poisson algebra, the reduced Poisson algebra of $C^\infty(\mathbb{M})$ by I , and is isomorphic to the Poisson algebra of functions $C^\infty(\mathbb{M}_{TT})$ on the reduced phase space.

4.5 The Hamiltonian as constraint and BRST

This technique of treating the Hamiltonian as another constraint in BRST will be used to apply BRST to Yang-Mills at the end of this section. It is useful for systems with secondary constraints. But before describing it for multiphase-space BRST in the following section, the phase-space version will be presented first in this section.

On extended phase space $M = \{(q^i, p_i, t, e)\}$, with symplectic form $\mathrm{d}q^i \wedge \mathrm{d}p_i + \mathrm{d}t \wedge \mathrm{d}e$, it was explained, in the subsection 2.1.4, 'Time Dependent Hamiltonians' , that the constraint $T^0 := H(q^i, p_i, t) - s$ generates the time evolution, and that the hypersurface M_{T^0}, the locus of $T^0 := H(q^i, p_i, t) - s = 0$, in extended phase space M, is foliated into unparametrized one-dimensional curves, which are the solutions to the Hamilton's equations for a Hamiltonian $H(q^i, p_i, t)$. The time is given by the t coordinate of the curve. The trajectories could be viewed as the orbits of a group, $G \equiv \mathbb{R}$ under addition, of symmetry variations where the variation parameter is time t. The constraint surface M_{T_0} is given by the solutions of $T_0 := (H - e) = 0$. For the constraint to be consistent with the orbits we require $\frac{\partial H}{\partial t} \approx \dot{e}$.

Assuming the solutions are complete, $-\infty < t < +\infty$, we can parametrize the different orbits by finding all the solutions to $H(q^i, p_i, 0) = e$. Here it can be seen that there is a solution for each point (q^i, p_i) in the phase space $\{(q^i, p_i)\}$. This latter is the reduced phase space $M//G \simeq M_{T_0}/\mathbb{R}$, with symplectic form $\mathrm{d}q^i \wedge \mathrm{d}p_i$.

It is possible to express this using the BRST formalism, where the starting Hamiltonian H_0 is zero, $H_0 \equiv 0$, and the physical Hamiltonian, H, appers as some terms (part of the grade-degree $(0,0)$ terms) in the gauge fixing expression: $H_{BRST} = H_0 + \delta_B \Psi = 0 + (H - e) + \text{ghost terms}$.

If there are no other symmetries to consider, there is thus a one dimensional trivial Lie algebra on extended phase space and the extra ghost-antighost grassmann odd canonical pair (c_0, τ), with the BRST observable $Q_0 = c_0 T_0 = c_0(H(q^i, p_i, t) - e)$. Clearly $\{Q_0, Q_0\} = 0$, therefore $\delta_B^2 \cdot = \{Q_0, \{Q_0, \cdot\}\} = 0$.

The global BRST variation δ_B on the super-phase space is

$$
\begin{aligned}
\delta_B q^i &= -\{Q_0, q^i\} = -c_0\{(H-e), q^i\} = c_0\frac{\partial H}{\partial p_i} \approx c_0\,\dot{q}^i \\
\delta_B p_i &= -\{Q_0, p_i\} = -c_0\{(H-e), p_i\} = -c_0\frac{\partial H}{\partial q^i} \approx c_0\,\dot{p}_i \\
\delta_B t &= -\{Q_0, t\} = -c_0\{(H-e), t\} = -c_0\frac{\partial t}{\partial t} = -c_0\,1 \approx -c_0\dot{t} \\
\delta_B e &= -\{Q_0, e\} = -c_0\{(H-e), e\} = -c_0\frac{\partial H}{\partial t} \approx -c_0\,\dot{e} \\
\delta_B c_0 &= -\{Q_0, c_0\} = 0 \\
\delta_B \tau &= -\{Q_0, \tau\} = -(H-e)\{c_0, \tau\} = -(H-e) \approx 0
\end{aligned}
\tag{4.54}
$$

The on-shell equations of motion on extended phase space for a Hamiltonian $(H-e)$ are shown above so that it can be seen, in these BRST variations, that the infinitesimal gauge variations corresponds to infinitesimal time evolution.

A suitable gauge fixing fermion is $\Psi = \dot{t}\tau$ so that the term that is added to the phase-space Lagrangian is:

$$
\begin{aligned}
\delta_B\Psi = -\{Q_0, \Psi\} = -\{c_0(H-e), \dot{t}\tau\} &= -(H-e)\dot{t}\{c_0, \tau\} + c_0\dot{\tau}\{(H-e), t\} \\
&= -(H-e)\dot{t} + c_0\dot{\tau} = -(H-e)\dot{t} - \dot{c}_0\tau
\end{aligned}
\tag{4.55}
$$

where the last equality employs integration by parts (inside the integral of a phase-space Lagrangian).

The super-phase-space Lagrangian is now

$$L = p_i\dot{q}^i - e\dot{t} + \tau\dot{c}_0 - (H-e)\dot{t} \tag{4.56}$$

and the super-phase-space BRST-fixed Hamiltonian is $(H-e)\dot{t}$. The ghost term decouple and the e coordinate cancels so that we are left with $H\dot{t} = H$ (because in the phase-space action integral $H\dot{t}\mathrm{d}\lambda = H\mathrm{d}t$, so t is the physical time corresponding to the physical Hamiltonian H). So the BRST method recreates the desired Hamiltonian for the system within the phase-space Lagrangian, $L = p_i\dot{q}^i - H$.

If there are other constraints T_a, we need to add extra terms to the BRST observable $Q = Q_0 + c^a T_a + \ldots$. In addition we need to have extra terms for the relations between the constraints and also the hamiltonian constraint: $\{T_a, T_b\} = -f^c_{ab}T_c$ and $\{T_a, T_0\} = -g^c_a T_c - h_a T_0$.

Assuming neither structure constants g^c_a or h_a are functions of q or p, there are then no higher relations and one can write the BRST observable so that $\{Q, Q\} = 0$:

$$Q = c_0(H-e) + c^a T_a - \frac{1}{2}f^a_{bc}c^b c^c \rho_a - \frac{1}{2}g^a_b c^b c_0 \rho_a - \frac{1}{2}h_a c^a c_0 \tau \tag{4.57}$$

4.5.1 Systems with primary and secondary constraints

In particular we will apply this to the case of a system with primary and secondary, but not tertiary, constraints as shown in subsection 3.6.2. This occurs in the case of a gauge variation which depends on a time-dependent parameter $\epsilon^a(t)$ and it's first derivative:

$$\delta_\epsilon(u^i(t)) = \epsilon^a(t)R^i_a(q(t)) + \partial_t\epsilon^a(t)S^i_a(q(t)) \tag{4.58}$$

The form of the gauge generator, for the infinitesimal gauge variation δ_r, in a system with primary and secondary constraints is (3.77) : $T_\epsilon = \partial_t\epsilon^a(t)T^{(1)}_a + \epsilon^a T^{(2)}_a = \partial_t\epsilon^a(t)T^{(1)}_a + \epsilon^a T^{(2)}_a$ where $T^{(1)}$ are the primary constraints and $T^{(2)}$ are the secondary constraints. The $\epsilon^a(x)$ are the parameters of the gauge variation δ_ϵ where ϵ takes values in the gauge algebra. The secondary constraint arises from the fact that the bracket of the primary constraint with the Hamiltonian is not zero:

$$\partial_t T^{(1)}_a \approx \{T^{(1)}_a, H\} \ = \ T^{(2)}_a \tag{4.59}$$

and the fact that there is no tertiary constraint (by assumption) is because the bracket of the secondary constraint with the Hamiltonian is zero:

$$\{T^{(2)}_a, H\} \ = \ 0 \tag{4.60}$$

If the generators is to be Poisson, we need the brackets of the constraints map to the Lie brackets of the gauge variation and the bracket between the constraints to be the following:

$$\begin{aligned}\{T^{(1)}_a, T^{(1)}_b\} &= 0\\ \{T^{(1)}_a, T^{(2)}_b\} &= -f^c_{ab}\ T^{(1)}_c\\ \{T^{(2)}_a, T^{(2)}_b\} &= f^c_{ab}T^{(2)}_c\end{aligned} \tag{4.61}$$

where f^c_{ab} are the structure constants of the Lie algebra of the gauge variation: $[\delta_{\epsilon_1}\ ,\ \delta_{\epsilon_2}] = \delta_{\epsilon_3}$, where $\epsilon_3(t) = [\epsilon_1(t)\ ,\ \epsilon_2(t)]$, in components: $\epsilon^c_3(x) = f^c_{ab}\epsilon^a_1(t)\epsilon^b_2(t)$.

BRST on systems with primary and secondary constraints and with the Hamiltonian as constraint

The gauge parameter and its time-derivative, $\epsilon^a(t), \partial_t\epsilon^a(t)$, are treated as two separate parameters and promoted to grassmann odd super-phase-space coordinates c^a_1, c^a_2 respectively, with canonically conjugate grassmann odd super-phase-space momentum coordinates ρ^1_a, ρ^2_a.

The BRST super-phase space is now $\{(t, e; q^i, p_i; c_0, \tau; c^a_1, \rho^1_a; c^a_2, \rho^2_a)\})$.

We are now in a position to write out the BRST observable using the non-zero structure constants in (4.59) and (4.61) above as coefficients in the higher order terms:

$$Q = c_0(H-e) + c_1^a T_a^{(1)} + c_2^a T_a^{(2)} - \frac{1}{2} f_{bc}^a c_1^b c_2^c \rho_a^1 + \frac{1}{2} f_{bc}^a c_2^b c_2^c \rho_a^2 + \frac{1}{2} c_1^a c_0 \rho_a^2 \qquad (4.62)$$

The extra terms involving ghost momenta are present to cancel brackets of the first three terms to ensure the property $\{Q, Q\} = 0$ which leads to $\delta_B^2 \cdot = \{Q, \{Q, \cdot\}\} = 0$.

4.6 The DDW Hamiltonian as constraint and multiphase-space BRST

4.6.1 Hybrid technique using spatial integrals

The multiphase-space DDW equations of motions are partial differential equations over space-time coordinate variables, and do not produce a foliation of multiphase space, unlike Hamilton's equations on phase space. This is a major difference between symplectic and multisymplectic mechanics. The DDW Hamiltonian does not generate a vector field multisymplectomorphism in the way that a Hamiltonian on phase space does. It would seem this fact prevents the use of the DDW Hamiltonian to generate a 'symmetry' where a BRST variation of a gauge fixing fermion Ψ could be used to produce the DDW Hamiltonian, $\delta\Psi = \mathcal{H} +$ ghost terms , in the same way as achieved above.

A way round this obstacle is the 'hybrid technique' which employs, as 'hybrid' observables, the *spatial integrals* of observables on multiphase space, so that the hybrid observables depend on time only rather than spacetime. In this way a time evolution 'symmetry' is obtained while the multiphase-space formalism is employed in the integrands.

We restrict to spacetime on a Minkowski space with orthonormal coordinates. We make a choice of a particular (but arbitrary) time coordinate x^α on the spacetime. (In fact we could choose x^α to be a space coordinate, because of the covariance of the system, but it is more familiar to discuss time evolution rather than space evolution.).

We integrate the $d-1$ form $H\mathrm{d}^{d-1}x_\alpha$ on multiphase space over the flat hypersurface $x^\beta = t$, where t is a constant. We call this the hypersurface S_β. Then,

$$\int_{S_\beta(t)} H\mathrm{d}^{d-1}x_\alpha = \delta_\alpha^\beta \int_{S_\alpha(t)} H\mathrm{d}^{d-1}x_\alpha \qquad \text{(no sum on repeated } \alpha \text{ indices)} \qquad (4.63)$$

and $X_{H\mathrm{d}x_\alpha} := -\{H\mathrm{d}x_\alpha, \cdot\} = -\{H, \cdot\}_\alpha = -\frac{\partial H}{\partial u^i}\frac{\partial}{\partial p_i^\alpha} + \frac{\partial H}{\partial p_i^\alpha}\frac{\partial}{\partial u^i}$

Let us define the multi-Poisson brackets of two spacetime $d-1$-forms:

$$\{H\mathrm{d}x_\alpha, J\mathrm{d}x_\gamma\} = \frac{1}{2}\,[\,-X_{H\mathrm{d}x_\alpha} \lrcorner\, \mathrm{d}_v(J\mathrm{d}x_\gamma) + X_{J\mathrm{d}x_\gamma} \lrcorner\, \mathrm{d}_v(H\mathrm{d}x_\alpha)\,]$$

$$= \frac{1}{2}\,[\,\{H, J\,\}_\gamma \mathrm{d}x_\alpha + \{H, J\,\}_\alpha \mathrm{d}x_\gamma\,] \tag{4.64}$$

where d_v is the vertical exterior derivative on the fibers over spacetime.

Then the spatial integral of these brackets is:

$$\int_{S_\beta(t)} \{H\mathrm{d}x_\alpha, J\mathrm{d}x_\gamma\} = \delta^\beta_\alpha \int_{S_\beta(t)} \frac{1}{2}\,\{H, J\,\}_\gamma \mathrm{d}x_\beta + \delta^\beta_\gamma \int_{S_\beta(t)} \frac{1}{2}\,\{H, J\,\}_\alpha \mathrm{d}x_\beta \tag{4.65}$$

(no sum on repeated indices)

In particular,

$$\int_{S_\alpha(t)} \{H\mathrm{d}x_\alpha, J\mathrm{d}x_\alpha\} = \int_{S_\alpha(t)} \{H, J\}_\alpha \mathrm{d}x_\alpha = \int_{S_\alpha(t)} [\,\frac{\partial H}{\partial u^i}\frac{\partial J}{\partial p^\alpha_i} - \frac{\partial H}{\partial p^\alpha_i}\frac{\partial J}{\partial u^i}\,]\,\mathrm{d}x_\alpha \tag{4.66}$$

(no sum on repeated indices)

We want to use the fact that the integral of the DeDonder-Weyl equation (DDW2) is:

$$\int_{S_\alpha(t)} \partial_\mu p^\mu_i \mathrm{d}^{d-1}x_\alpha \approx \int_{S_\alpha(t)} -\frac{\partial H}{\partial u^i}\mathrm{d}^{d-1}x_\alpha \tag{4.67}$$

The right hand side is

$$\int_{S_\alpha(t)} -\frac{\partial H}{\partial u^i}\,\mathrm{d}^{d-1}x_\alpha = \int_{S_\alpha(t)} -\{H, p^\mu_i \mathrm{d}x_\mu\}\,\mathrm{d}^{d-1}x_\alpha = \int_{S_\alpha(t)} -\{H\mathrm{d}x_\alpha, p^\alpha_i \mathrm{d}x_\alpha\} \tag{4.68}$$

(no sum on repeated α indices). The left hand side is

$$\int_{S_\alpha(t)} \partial_\mu p^\mu_i \mathrm{d}^{d-1}x_\alpha = \int_{S_\alpha(t)} \partial_\alpha p^\alpha_i\,\mathrm{d}^{d-1}x_\alpha = \partial_\alpha \int_{S_\alpha(t)} p^\alpha_i \mathrm{d}^{d-1}x_\alpha = \frac{\mathrm{d}}{\mathrm{d}t}\int_{S_\alpha(t)} p^\alpha_i \mathrm{d}^{d-1}x_\alpha \tag{4.69}$$

(no sum on repeated α indices), because we assume that the field multimomenta are zero at the spatial boundary.

We also have the integral of DDW1:

$$\int_{S_\alpha(t)} \partial_\mu u^i\,\mathrm{d}^{d-1}x_\alpha \approx \int_{S_\alpha(t)} \frac{\partial H}{\partial p^\mu_i}\,\mathrm{d}^{d-1}x_\alpha \tag{4.70}$$

The right hand side is

$$\int_{S_\alpha(t)} \frac{\partial H}{\partial p^\mu_i}\,\mathrm{d}^{d-1}x_\alpha = \int_{S_\alpha(t)} -\{H, u^i\}_\mu\,\mathrm{d}^{d-1}x_\alpha = \int_{S_\alpha(t)} -\{H\mathrm{d}x_\mu, u^i\}\,\mathrm{d}^{d-1}x_\alpha$$

$$= \delta^\alpha_\mu \int_{S_\alpha(t)} -\{H\mathrm{d}x_\alpha, u^i\}\,\mathrm{d}^{d-1}x_\alpha = \delta^\alpha_\mu \int_{S_\alpha(t)} -\{H\mathrm{d}x_\alpha, u^i \mathrm{d}^{d-1}x_\alpha\} \tag{4.71}$$

The left hand side is

$$\int_{S_\alpha(t)} \partial_\mu u^i \ \mathrm{d}^{d-1}x_\alpha = \delta^\alpha_\mu \int_{S_\alpha(t)} \partial_\mu u^i \ \mathrm{d}^{d-1}x_\alpha = \delta^\alpha_\mu \int_{S_\alpha(t)} \partial_\alpha u^i \ \mathrm{d}^{d-1}x_\alpha = \delta^\alpha_\mu \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_\alpha(t)} u^i \ \mathrm{d}^{d-1}x_\alpha \tag{4.72}$$

(no sum on repeated indices), because we assume that the fields u^i are zero at the spatial boundary.

So (4.70) can be written as

$$\delta^\alpha_\mu \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_\alpha(t)} u^i \ \mathrm{d}^{d-1}x_\alpha \approx \delta^\alpha_\mu \int_{S_\alpha(t)} -\{H\mathrm{d}x_\alpha, u^i \mathrm{d}^{d-1}x_\alpha\}$$

$$= \int_{S_\alpha(t)} -\{H\mathrm{d}x_\alpha, u^i \mathrm{d}^{d-1}x_\mu\} = \int_{S_\alpha(t)} -\{H\mathrm{d}x_\mu, u^i \mathrm{d}^{d-1}x_\alpha\} \tag{4.73}$$

(no sum on repeated indices). This can be easily generalized to

$$\delta^\alpha_\mu \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_\alpha(t)} f(u) \ \mathrm{d}^{d-1}x_\alpha \approx \delta^\alpha_\mu \int_{S_\alpha(t)} -\{H\mathrm{d}x_\alpha, \ f(u) \ \mathrm{d}^{d-1}x_\alpha\}$$

$$= \int_{S_\alpha(t)} -\{H\mathrm{d}x_\alpha, \ f(u) \ \mathrm{d}^{d-1}x_\mu\} = \int_{S_\alpha(t)} -\{H\mathrm{d}x_\mu, \ f(u) \ \mathrm{d}^{d-1}x_\alpha\} \tag{4.74}$$

(no sum on repeated indices)

The most general $d-1$ -form observables (apart from the DDW Hamiltonian $d-1$ -form $H\mathrm{d}^{d-1}x_\alpha$) we will consider on multiphase space will have terms of the form $O(u^i, p^\alpha_i) = f^i(u) p^\alpha_i \mathrm{d}^{d-1}x_\alpha$ or$f(u)\mathrm{d}^{d-1}x_\alpha$.

The DDW equation of motion for this is:

$$\{f^i(u)p^\alpha_i \mathrm{d}^{d-1}x_\alpha, H\} \ = \ f^i_{,j}(u)p^\alpha_i\{u^j \mathrm{d}^{d-1}x_\alpha, H\} \ + \ f^i(u)\{p^\alpha_i \mathrm{d}^{d-1}x_\alpha, H\}$$

$$= \ f^i_{,j}(u)p^\alpha_i \frac{\partial H}{\partial p^\alpha_j} \ - \ f^i(u)\frac{\partial H}{\partial u^i} \ \approx \ \partial_\alpha(f^i(u(x))p^\alpha_i(x)) \tag{4.75}$$

The spatial integral of this is:

$$\frac{\mathrm{d}}{\mathrm{d}t} \int_{S_\alpha(t)} (f^i(u(x))p^\alpha_i(x)) \ \mathrm{d}^{d-1}x_\alpha = \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_\alpha(t)} (f^i(u(x))p^\nu_i(x)) \ \mathrm{d}^{d-1}x_\nu$$

$$= \int_{S_\alpha(t)} \partial_\nu (f^i(u(x))p^\nu_i(x))\mathrm{d}^{d-1}x_\alpha$$

$$\approx \int_{S_\alpha(t)} \{f^i(u)p^\nu_i \mathrm{d}^{d-1}x_\nu, H\} \ \mathrm{d}^{d-1}x_\alpha \ = \ \int_{S_\alpha(t)} \{f^i(u)p^\alpha_i \mathrm{d}^{d-1}x_\alpha, H\mathrm{d}^{d-1}x_\alpha\} \tag{4.76}$$

(no sum on repeated α indices), because we assume that the field multimomenta are zero at the spatial boundary.

The corresponding spatial integral on *extended* multiphase space is:

$$\int_{S_\alpha(t)} \{f^i(u)p^\alpha_i \mathrm{d}^{d-1}x_\alpha, (H-e)\mathrm{d}^{d-1}x_\alpha\}$$

$$= \int_{S_\alpha(t)} [f^i_{,j}(u) p^\nu_i \frac{\partial H}{\partial p^\nu_j} \; - \; f^i(u) \frac{\partial H}{\partial u^i} - \partial_\nu (f^i(u) p^\nu_i)] \; \mathrm{d}^{d-1} x_\alpha$$

$$= \int_{S_\alpha(t)} \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, (H - e) \} \; \mathrm{d}^{d-1} x_\alpha$$

$$= \int_{S_\alpha(t)} [f^i_{,j}(u) p^\nu_i \frac{\partial H}{\partial p^\nu_j} \; - \; f^i(u) \frac{\partial H}{\partial u^i} \;] \mathrm{d}^{d-1} x_\alpha - \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_\alpha(t)} (f^i(u) p^\alpha_i) \; \mathrm{d}^{d-1} x_\alpha$$

$$\approx \int_{S_\alpha(t)} \partial_\nu (f^i(u) p^\nu_i) \mathrm{d}^{d-1} x_\alpha - \int_{S_\alpha(t)} \partial_\nu (f^i(u) p^\nu_i) \mathrm{d}^{d-1} x_\alpha \; = \; 0 \tag{4.77}$$

(no sum on repeated α indices). We used the integrated equation of motion in the last $\approx$.

We need the gauge variation of observables to be compatible with trajectories which satisfy the integrated equations of motion: $(-\delta_\epsilon \frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t} \delta_\epsilon) O = 0$.

Calculating the first term,

$$\delta_\epsilon \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_\alpha(t)} (f^i(u) p^\alpha_i) \; \mathrm{d}^{d-1} x_\alpha = \delta_\epsilon \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_\alpha(t)} (f^i(u) p^\nu_i) \; \mathrm{d}^{d-1} x_\nu = \delta_\epsilon \int_{S_\alpha(t)} \partial_\nu (f^i(u) p^\nu_i) \mathrm{d}^{d-1} x_\alpha$$

$$\approx \; \delta_\epsilon \int_{S_\alpha(t)} \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, H \} \; \mathrm{d}^{d-1} x_\alpha \; = \; \int_{S_\alpha(t)} \delta_\epsilon \{ f^i(u) p^\alpha_i \mathrm{d}^{d-1} x_\alpha, H \} \; \mathrm{d}^{d-1} x_\alpha$$

$$= \; - \int_{S_\alpha(t)} \{ G, \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, H \} \} \tag{4.78}$$

(no sum on repeated α indices), where G is the generator of the gauge transformation, $\delta_\epsilon \cdot = -\{G . \cdot\}$.

Calculating the second term

$$\frac{\mathrm{d}}{\mathrm{d}t} \delta_\epsilon \int_{S_\alpha(t)} (f^i(u) p^\alpha_i) \; \mathrm{d}^{d-1} x_\alpha = \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_\alpha(t)} \delta_\epsilon (f^i(u) p^\nu_i) \; \mathrm{d}^{d-1} x_\nu$$

$$= \; \frac{\mathrm{d}}{\mathrm{d}t} \int_{S_\alpha(t)} \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, G \} \; \mathrm{d}^{d-1} x_\alpha \; = \int_{S_\alpha(t)} \frac{\partial}{\partial x^\alpha} \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, G \} \; \mathrm{d}^{d-1} x_\alpha$$

$$= \int_{S_\alpha(t)} \frac{\partial}{\partial x^\alpha} \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, G \} \; \mathrm{d}^{d-1} x_\alpha$$

$$\approx \int_{S_\alpha(t)} \{ \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, G \}, H \} \; + \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, \frac{\partial}{\partial x^\alpha} G \} \mathrm{d}^{d-1} x_\alpha \tag{4.79}$$

(no sum on repeated α indices)

Subtracting the above the above terms, we want $(-\delta_\epsilon \frac{\mathrm{d}}{\mathrm{d}t} + \frac{\mathrm{d}}{\mathrm{d}t} \delta_\epsilon) O = 0$ resulting in:

$$\int_{S_\alpha(t)} [\{ G, \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, H \} \} + \{ \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, G \}, H \} + \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, \frac{\partial}{\partial x^\alpha} G \}] \; \mathrm{d}^{d-1} x_\alpha$$

$$= \; \int_{S_\alpha(t)} [\{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, \{ G, H \} \} \; + + \; \{ f^i(u) p^\nu_i \mathrm{d}^{d-1} x_\nu, \frac{\partial}{\partial x^\alpha} G \}] \; \mathrm{d}^{d-1} x_\alpha \; = \; 0 \tag{4.80}$$

Using the Jacobi identity (which holds in the α direction) to go to the second line. Thus we need $\{ \; O(u,p) \; , \; \{G, H\} + \frac{\partial}{\partial x^\alpha} G \; \} = 0$.

4.6.2 The multiphase-space BRST construction for time evolution

We work on the multiphase space $M = \{(e, x^\mu; u^i, p_i^\mu)\}$, with multisymplectic form $(\mathrm{d}u^i \wedge \mathrm{d}p_i^\mu + \mathrm{d}e \wedge \mathrm{d}x^\mu) \wedge \mathrm{d}x_\mu$. We can view the 'DDW energy' coordinate e in multiphase space having as canonical multimomenta the spacetime coordinates x^μ.

It was shown in (4.77) the previous subsection the constraint $T_\alpha^0 := (\mathcal{H}(u^i, p_i^\mu, t) - e)\mathrm{d}^{d-1}x_\alpha$, where $\mathcal{H}$ is the DDW Hamiltonian, generates the time evolution on the hybrid observables:

$$\int_{S_\alpha(t)} \{O, T_\alpha^0\} \;=\; \int_{S_\alpha(t)} \{f^i(u)p_i^\nu \mathrm{d}^{d-1}x_\nu, (\mathcal{H} - e)\mathrm{d}^{d-1}x_\alpha\}$$

$$= \int_{S_\alpha(t)} [f^i_{,j}(u)p_i^\nu \frac{\partial \mathcal{H}}{\partial p_j^\nu} \;-\; f^i(u)\frac{\partial \mathcal{H}}{\partial u^i}\,]\mathrm{d}^{d-1}x_\alpha - \frac{\mathrm{d}}{\mathrm{d}t}\int_{S_\alpha(t)} (f^i(u)p_i^\alpha)\;\mathrm{d}^{d-1}x_\alpha \approx 0 \qquad (4.81)$$

(no sum on repeated α indices).

The trajectories could be viewed as the orbits of a group, $\mathbb{R}$ under addition, of symmetry variations where the variation parameter is time $t = x^\alpha$. The constraint surface M_{T_0} is given by the solutions of $(\mathcal{H} - e) = 0$. For the constraint to be consistent with the orbits we require $\frac{\partial \mathcal{H}}{\partial x^\mu} \approx \partial_\mu e$. The different trajectories are parametrized by the initial conditions at time $t = x^\alpha = t_0$. The initial conditions are the values of the u^i and p_i^α on the points of the spatial hypersurface $x^\alpha = t_0$ in Minkowski space M^d. Note that the p_i^α are the conventional momenta because we have chosen α to be the time direction. The trajectories are distinct because it is well known that for a field theory this characterization on a spatial hypersurface determines a unique trajectory because it is a point in the phase space of the system which can be described by first order ordinary differential equations in time.

It is possible express this using the BRST formalism, where the starting DDW Hamiltonian $\mathcal{H}_0$ is zero and the gauge fixing will gives the BRST Hamiltonian $\mathcal{H}_B = \mathcal{H}_0 + \delta_B\Psi = 0 + (\mathcal{H} - e)x^\alpha +$ ghost terms .

We show this using the hybrid technique.

If there are no other symmetries to consider, there is thus a one dimensional trivial Lie algebra on extended phase space and the extra ghost-antighost grassmann odd canonical pair (τ, c_0^α) on multiphase space, with the BRST observable $Q_0 = c_0^\alpha(\mathcal{H}(u^i, p_i^\mu, x) - e)\mathrm{d}^{d-1}x_\alpha$. Clearly $\{Q_0, Q_0\} = 0$, therefore $\delta_B^2 \cdot = \{Q_0, \{Q_0, \cdot\}\} = 0$. We assume these are integrands of the integral over the spatial section $S_\alpha(t)$:

$$\int_{S_\alpha(t)} \delta_B^2(f^i(u)p_i^\nu \mathrm{d}^{d-1}x_\nu)\mathrm{d}^{d-1}x_\alpha = \int_{S_\alpha(t)} \{Q_0, \{Q_0, f^i(u)p_i^\nu \mathrm{d}^{d-1}x_\nu\}\}\mathrm{d}^{d-1}x_\alpha = 0 \qquad (4.82)$$

The pointwise multibracket calculation of the the variation δ_B above on the super-phase space is

$$
\begin{aligned}
\delta_B u^i &= -\{Q_0, u^i\} = -c_0^\mu \{(\mathcal{H}-e), u^i\}_\mu = c_0^\mu \frac{\partial \mathcal{H}}{\partial p_i^\mu} \approx c_0^\mu\ \partial_\mu u^i \\
\delta_B(p_i^\mu \mathrm{d}x_\mu) &= -\{Q_0, p_i^\mu \mathrm{d}x_\mu\ \} = -c_0^\nu \mathrm{d}x_\nu \{(\mathcal{H}-e), p_i^\mu \mathrm{d}x_\mu\ \} = -c_0^\nu \mathrm{d}x_\nu \frac{\partial \mathcal{H}}{\partial u^i} \approx c_0^\nu \mathrm{d}x_\nu\ \partial_\mu p_i^\mu \\
\delta_B(x^\mu \mathrm{d}x_\mu) &= -\{Q_0, x^\mu \mathrm{d}x_\mu\} = -c_0^\nu \mathrm{d}x_\nu \{(\mathcal{H}-e), x^\mu \mathrm{d}x_\mu\} = c_0^\nu \mathrm{d}x_\nu\, d \approx c_0^\nu \mathrm{d}x_\nu\ \partial_\mu x^\mu \\
\delta_B e &= -\{Q_0, e\} = -c_0^\nu \mathrm{d}x_\nu \{(\mathcal{H}-e), e\} = -c_0^\nu\ \mathrm{d}x_\nu \frac{\partial \mathcal{H}}{\partial x^\mu} \approx -c_0^\nu \mathrm{d}x_\nu\ \partial_\mu e \\
\delta_B(c_0^\nu \mathrm{d}x_\nu) &= -\{Q_0, c_0^\nu \mathrm{d}x_\nu\} = 0 \\
\delta_B \tau &= -\{Q_0, \tau\} = -(\mathcal{H}-e)\{c_0^\nu \mathrm{d}x_\nu, \tau\} = -(\mathcal{H}-e) \approx 0
\end{aligned}
\tag{4.83}
$$

The spacetime derivative of x^μ above: $\partial_\mu x^\mu$, reflects the fact that the selection of the section ∂_μ through extended multiphase space, which obeys the equation of motion (indicated by the use of $\approx$), acting on the coordinates x^μ of extended multiphase space, are consistent, $\partial_\mu x^\mu \approx d$ in d-dimensional spacetime.

The on-shell equations of motion (indicated by the '$\approx$') on extended multiphase space for a DDW Hamiltonian $(\mathcal{H}-e)$ are included above so that it can be seen, in these BRST variations, that the infinitesimal gauge variations corresponds to the DDW infinitesimal equation of motion (propagation in time).

4.6.3 Gauge fixing for time evolution

A suitable BRST gauge fixing fermion is $\Psi = \ \tau x^\alpha \mathrm{d}x_\alpha$ (no sum on the α index) so that the term that is added to the multiphase-space Lagrangian is:

$$
\begin{aligned}
\delta_B \Psi &= -\{Q_0, \Psi\} = -\{c_0^\mu \mathrm{d}x_\mu (\mathcal{H}-e), x^\alpha \tau \mathrm{d}x_\alpha\} \\
&= -(\mathcal{H}-e) x^\alpha \mathrm{d}x_\alpha \{c_0^\mu \mathrm{d}x_\mu, \tau\} + c_0^\mu \mathrm{d}x_\mu \tau \{(\mathcal{H}-e), x^\alpha \mathrm{d}x_\alpha\} \\
&= -(\mathcal{H}-e) x^\alpha \mathrm{d}x_\alpha + c_0^\mu \mathrm{d}x_\mu \tau
\end{aligned}
\tag{4.84}
$$

where the integration over a spatial slice $S_\alpha(t)$ needs to be assumed because $x^\alpha = t$ needs to be a constant.

The super-multi-phase space BRST gauge-fixed Hamiltonian is $(\mathcal{H}-e)x^\alpha \mathrm{d}x_\alpha$. The ghost term dynamically decouples from the physical variables. So the BRST method recreates the desired DDW Hamiltonian for the system. The x^α factor in the spatial integration is the time and is the time increment in the full integral over spacetime when it is present in the multiphase-space Lagrangian.

If there are other constraints T_a, we need to add extra terms to the BRST observable $Q =$

$Q_0 + c^a T_a + \dots$. In addition, we need to have extra terms for the relations between the constraints: $\{T_a, T_b\} = -f^c_{ab} T_c$ and $\{T_a, T_0\} = -g^c_a T_c - h_a T_0$, to ensure $\{Q, Q\} = 0$.

Assuming neither structure constants g^c_a or h_a are functions of multiphase-space coordinates x^μ, q , p, t or e, then there are no higher relations and one can write the BRST function so that $\{Q, Q\} = 0$:

So $Q = c^\alpha_0 (H - e)\mathrm{d}x_\alpha + c^a T^\alpha_a \mathrm{d}x_\alpha - \frac{1}{2} f^a_{bc} c^b c^c \rho^\alpha_a \mathrm{d}x_\alpha - \frac{1}{2} g^a_b c^b c^\alpha_0 \rho^\alpha_a \mathrm{d}x_\alpha - \frac{1}{2} h_a c^a c^\alpha_0 \tau \mathrm{d}x_\alpha$.

If the structure constants g^c_a or h_a *are* functions of the extended phase-space coordinates, then brackets like $\{g^c_a, \cdot\}$ or $\{h_a, \cdot\}$ need to be calculated and included, as well as brackets with the results of these brackets, until one can ensure $\{Q, Q\} = 0$ and we will have higher ghost degree terms, reflecting the higher relations of the constraint algebra.

Systems with primary and secondary constraints

This topic was introduced in section 3.6.2. This will now be applied to an example, Yang-Mills, to obtain the BRST observable.

When action is invariant under the infinitesimal gauge variation of the field $u(x)$ is given by the parameters $\epsilon^a(x)$ and the spacetime derivatives $\partial_\mu \epsilon^a(x)$:

$$\delta_\epsilon(u^i(x)) = \partial_\mu \epsilon^a(x) S^{i\mu}_a(u(x)) + \epsilon^a(x) R^i_a(u(x)) \tag{4.85}$$

The form of the gauge generator, for the infinitesimal gauge variation δ_ϵ, in a system with primary and secondary constraints is : $T_\epsilon = \partial_\nu \epsilon^a(x) T^{(1)\nu}_a + \epsilon^a T^{(2)}_a = \partial_\nu \epsilon^a(x) T^{(1)\nu\mu}_a \mathrm{d}x_\mu + \epsilon^a T^{(2)\mu}_a \mathrm{d}x_\mu$ where $T^{(1)\nu\mu}$ are the primary constraints and $T^{(2)\mu}$ are the secondary constraints. The $\epsilon^a(x)$ are the parameters of the gauge variation δ_ϵ where ϵ takes values in the gauge algebra.

The secondary constraint arises from the fact that the bracket of the primary constraint with the DDW Hamiltonian is not zero:

$$\partial_\nu \mathrm{T}^{(1)\alpha\nu}_a \approx \{T^{(1)\alpha}_a, \mathcal{H}\} \; = \; T^{(2)\alpha}_a \tag{4.86}$$

and the fact that there is no tertiary constraint is because the bracket of the secondary constraint with the DDW Hamiltonian is zero:

$$\{T^{(2)}_a, \mathcal{H}\} \; = \; 0 \tag{4.87}$$

If the gauge generator is to be Poisson, we require the brackets of the constraints map to the Lie brackets of the gauge variation and the multi-bracket between the constraints to be the following:

$$\begin{aligned}\{T_a^{(1)\alpha}, T_b^{(1)\beta}\} &= 0 \\ \{T_a^{(1)\alpha}, T_b^{(2)}\} &= -f_{ab}^c\, T_c^{(1)\alpha} \\ \{T_a^{(2)}, T_b^{(2)}\} &= f_{ab}^c T_c^{(2)}\end{aligned} \tag{4.88}$$

where f_{ab}^c are the structure constants of the Lie algebra of the gauge variation: $[\delta_{\epsilon_1}\delta_{\epsilon_2}] = \delta_{\epsilon_3}$ $\epsilon_3(x) = [\epsilon_1(x), \epsilon_2(x)]$, $\epsilon_3^c(x) = f_{ab}^c \epsilon_1^a(x)\epsilon_2^b(x)$.

BRST for systems with primary and secondary constraints

The gauge parameter and its spacetime-derivative, $\partial_\mu, \epsilon^a(x)\epsilon^a(x)$, are treated as two separate parameters and promoted to grassmann odd super-multiphase-space coordinates $c_{1\nu}^a, c_2^a$ respectively, with canonically conjugate grassmann odd super- multiphase-space momentum coordinates $\rho_a^{1\nu\mu}, \rho_a^{2\mu}$.

$$T_\epsilon = \partial_\nu \epsilon^a(x) T_a^{(1)\nu} + \epsilon^a T_a^{(2)} = \partial_\nu \epsilon^a(x) T_a^{(1)\nu\mu} \mathrm{d}x_\mu + \epsilon^a T_a^{(2)\mu} \mathrm{d}x_\mu \tag{4.89}$$

The BRST super-multiphase space is now $\{(x^\mu, u^i, e, p_i^\mu, c_0^\mu, c_{1\nu}^a, c_2^a, \tau, \rho_a^{1\nu\mu}, \rho_a^{2\mu})\}$.

We are now in a position to write out the BRST observable using the non-zero structure constants in (4.86) and (4.88) above as coefficients in the higher order terms.

$$Q = c_0^\alpha (H - e)\mathrm{d}x_\alpha + c_{1\nu}^a T_a^{(1)\nu\alpha} \mathrm{d}x_\alpha + c_2^a T_a^{(2)\alpha} \mathrm{d}x_\alpha - \frac{1}{2} f_{bc}^a c_{1\nu}^b c_2^c \rho_a^{1\nu\alpha} \mathrm{d}x_\alpha + \frac{1}{2} f_{bc}^a c_2^b c_2^c \rho_a^{2\alpha} \mathrm{d}x_\alpha + \frac{1}{2} c_{1\mu}^a c_0^\alpha \rho_a^{2\mu} \mathrm{d}x_\alpha \tag{4.90}$$

(No sum on the α indices).

Or in slightly more compact notation:

$$Q = c_0^\alpha (H - e)\mathrm{d}x_\alpha + c_{1\nu}^a T_a^{(1)\nu} + c_2^a T_a^{(2)} - \frac{1}{2} f_{bc}^a c_{1\nu}^b c_2^c \rho_a^{1\nu} + \frac{1}{2} f_{bc}^a c_2^b c_2^c \rho_a^2 - \frac{1}{2} c_1^a \lrcorner\, \rho_a^2 c_0^\alpha \mathrm{d}x_\alpha \tag{4.91}$$

Q generates the variations:

$$\delta_Q u^i = -\{Q, u^i\} = c_0^\alpha \mathrm{d}x_\alpha \{\mathcal{H} - e, u^i\} + c_{1\nu}^a S_a^{i\nu}(u) + c_2^a R_a^i(u) \approx c_0^\alpha \partial_\alpha u^i + c_{1\nu}^a S_a^{i\nu}(u) + c_2^a R_a^i(u) \tag{4.92}$$

$$\begin{aligned}\delta_Q(p_i^\alpha \mathrm{d}x_\alpha) = -\{Q, p_i^\alpha \mathrm{d}x_\alpha\} &= -c_0^\alpha \mathrm{d}x_\alpha \{\mathcal{H}, p_i^\alpha \mathrm{d}x_\alpha\} + c_{1\nu}^a \partial_j S_a^{i\nu} p_j^\alpha \mathrm{d}x_\alpha) + c_2^a \partial_j R_a^i p_j^\alpha \mathrm{d}x_\alpha \\ &\approx c_0^\beta \mathrm{d}x_\beta \partial_\alpha p_i^\alpha + c_{1\nu}^a \partial_j S_a^{i\nu} p_j^\alpha \mathrm{d}x_\alpha) + c_2^a \partial_j R_a^i p_j^\alpha \mathrm{d}x_\alpha\end{aligned} \tag{4.93}$$

$$\delta_Q e = -\{Q, e\} = c_0^\alpha \frac{\partial Q}{\partial x^\alpha} = c_0^\alpha \frac{\partial \mathcal{H}}{\partial x^\alpha} \tag{4.94}$$

The last equality assumes only $\mathcal{H}$ (and not T , f , R or S) can explicitly depend on x.

$$\delta_Q(x^\mu \mathrm{d}x_\mu) = -\{Q, x^\mu \mathrm{d}x_\mu\} = \{e, x^\mu \mathrm{d}x_\mu\} = 1 \approx \frac{1}{d}\partial_\mu x^\mu \tag{4.95}$$

$$\delta_Q\tau = -\{Q,\tau\} = -(H-e) - \frac{1}{2}c^a_{1\beta}\rho^{2\beta}_a \tag{4.96}$$

$$\delta_Q(c^\mu_0 \mathrm{d}x_\mu) = -\{Q, c^\mu_0 \mathrm{d}x_\mu\} = 0 \tag{4.97}$$

$$\delta_Q c^a_{1\nu} = -\{Q, c^a_{1\nu}\} = \frac{1}{2}f^a_{bc}c^b_{1\nu}c^c_2 \tag{4.98}$$

$$\delta_Q(\rho^{1\nu\beta}_a \mathrm{d}x_\beta) = -\{Q, \rho^{1\nu\beta}_a \mathrm{d}x_\beta\} = -T^{(1)\nu\beta}_a \mathrm{d}x_\beta + \frac{1}{2}f^b_{ac}c^c_2\rho^{1\nu\beta}_b \mathrm{d}x_\beta \tag{4.99}$$

Note free index ν.

$$\delta_Q c^a_2 = -\{Q, c^a_2\} = -\frac{1}{2}f^a_{bc}c^b_2 c^c_2 - \frac{1}{2}c^a_{1\beta}c^\beta_0 \tag{4.100}$$

$$\delta_Q(\rho^{2\beta}_a \mathrm{d}x_\beta) = -\{Q, \rho^{2\beta}_a \mathrm{d}x_\beta\} = -T^{(2)\beta}_a \mathrm{d}x_\beta - \frac{1}{2}f^c_{ba}c^b_{1\nu}\rho^{1\nu\beta}_c \mathrm{d}x_\beta + f^c_{ba}c^b_2\rho^{2\beta}_c \mathrm{d}x_\beta \tag{4.101}$$

4.6.4 Yang-Mills

Yang Mills is a system with primary and secondary constraints where the above technique should be applicable. Repeating the constraint algebra (3.110) for Yang-Mills given in section 3.8:

$$\{T^{1\alpha}_a, T^{1\beta}_b\} = 0 \tag{4.102}$$

$$\{T^{1\alpha}_a, T^2_b\} = -gf^c_{ab}\, T^{1\alpha}_c \tag{4.103}$$

$$\{T^2_a, T^2_b\} = gf^c_{ab}T^2_c \tag{4.104}$$

$$\{T^{1\alpha}_a, \mathcal{H}\} = T^{2\alpha}_c \tag{4.105}$$

$$\{T^2_a, \mathcal{H}\} = 0 \tag{4.106}$$

where $T^{1\alpha\mu}_a \mathrm{d}x_\mu = S^{i\alpha}_{a\beta}p^{\beta\mu}_i \mathrm{d}x_\mu = p^{\alpha\mu}_a \mathrm{d}x_\mu$ and $T^2_b = R^i_{b\alpha}p^{\alpha\mu}_i \mathrm{d}^{d-1}x_\mu = (gp^{[\alpha\mu]}_a f^a_{bc}A^c_\alpha)\mathrm{d}^{d-1}x_\mu$.

Substituting these into the general expression for Q in the previous section gives the following BRST observable for the specific case of Yang-Mills:

$$Q_{YM} = [c^\alpha_0(\frac{1}{2}p^{[\mu\nu]}_a p^a_{[\mu\nu]} + gp^{[\mu\nu]}_a f^a_{bc}A^b_\mu A^c_\nu + p^{(\mu\nu)}_a \partial_\mu A^a_\nu - e)\mathrm{d}x_\alpha$$

$$+c^a_{1\nu}p^{\nu\alpha}_a \mathrm{d}x_\alpha + c^a_2 gf^b_{ac}p^{\mu\alpha}_b A^c_\mu \mathrm{d}x_\alpha - \frac{1}{2}gf^a_{bc}c^b_{1\nu}c^c_2\rho^{1\nu\alpha}_a \mathrm{d}x_\alpha + \frac{1}{2}gf^a_{bc}c^b_2 c^c_2\rho^{2\alpha}_a \mathrm{d}x_\alpha + \frac{1}{2}c^a_{1\mu}c^\alpha_0\rho^{2\mu}_a \mathrm{d}x_\alpha \tag{4.107}$$

4.7 Multiphase-space BRST examples

Abelian 4.7.1 and non-Abelian 4.7.2 Yang-Mills theory are presented in such a way that the comparison with the analogous example of conventional BRST applied to the electromagnetic field in section 4.3.1 can be readily made.

4.7.1 The electromagnetic field

The example of the electromagnetic field follows from the exposition of electromagnetism in the multiphase-space formalism in section D.2. The construction is in two steps: initially we write the (multiphase-space) Lagrangian density on the space of dynamical variables (the pre-super-multiphase space) $p\mathcal{B} = \{(x^\mu, A_\alpha, p^{\alpha\mu}, c, \bar{c}^{\mu\nu}, \Pi^\nu_{\bar{c}}, b^{\mu\nu})\}$ and then we reduce to the super-multiphase space $\mathcal{B} = \{(x^\mu, A_\alpha, p^{\alpha\mu}, c, \rho^\mu)\}$.

To apply the super multiphase space BRST formalism to this model we replace, in the gauge variation, the gauge parameter function $f(x)$ by a grassmannian field $c(x)$ (called a ghost field) which we will adjoin to this model as a dynamical variable, together with additional grassmanian field $\bar{c}^{\mu\nu}(x)$ (anti-ghost) and commuting field $b^{\mu\nu}(x)$ (Nakanishi-Lautrup field) which are both symmetric in $\mu\nu$. The objective is to add a gauge fixing term $b^{\mu\nu}\partial_\mu A_\nu - \frac{\xi}{2} b^{\mu\nu} b_{\mu\nu}$ (in such a way that $b^{\mu\nu} \approx p^{(\mu\kappa)}$ on shell) to the first order Lagrangian (D.26) in such a way that it remains invariant under the nilpotent global BRST variation δ_B on the pre-super-multiphase space $B = \{(x^\mu, A_\alpha, p^{\alpha\mu}, c, \bar{c}^{\mu\nu}, \Pi^\nu_{\bar{c}}, b^{\mu\nu})\}$, defined by

$$\begin{aligned}
\delta_B A_\alpha &= \partial_\alpha c \\
\delta_B p^{\alpha\mu} &= 0 \\
\delta_B c &= 0 \\
\delta_B \bar{c}^{\mu\nu} &= b^{\mu\nu} \\
\delta_B b^{\mu\nu} &= 0 \\
\delta_B x^\mu &= 0
\end{aligned} \tag{4.108}$$

With this variation it is clear that the original Lagrangian density (D.19) and the multiphase-space Lagrangian density (D.26) for the electromagnetic field, as functions on super-multiphase space, remain invariant under this variation, because we have simply replaced the gauge variation parameter $f(x)$ by $c(x)$. Then adding a term of the form $\delta_B \Psi$, the multiphase-space Lagrangian density becomes the multiphase-space BRST first order action (4.109) and will still be invariant because of the $\delta_B \delta_B = 0$ nilpotent property of the variation above. A suitable gauge fixing fermion is $\Psi = \bar{c}^{\mu\nu}(\partial_\mu A_\nu - \frac{\xi}{2} b_{\mu\nu})$, where ξ is a constant real number factor, which results in $\delta_B \Psi = b^{\mu\nu}\partial_\mu A_\nu - \frac{\xi}{2} b^{\mu\nu} b_{\mu\nu} - \bar{c}^{\mu\nu}\partial_\mu\partial_\nu c$, which is the desired gauge fixing term plus a ghost term.

The multiphase-space BRST first order action is then:

$$S_\Psi = \int_{\Gamma pB} (p^{\mu\nu}\partial_\mu A_\nu - H + \delta_B \Psi)\mathrm{d}^d x =$$

$$\int_{\Gamma pB} (p^{\mu\nu}\partial_\mu A_\nu - \frac{1}{2} p^{[\mu\nu]} p_{[\mu\nu]} - p^{(\mu\nu)}\partial_\mu A_\nu + (b^{\mu\nu}\partial_\mu A_\nu - \frac{\xi}{2} b^{\mu\nu} b_{\mu\nu} - \bar{c}^{\mu\nu}\partial_\mu\partial_\nu c)\ )\mathrm{d}^d x =$$

$$\int_{\Gamma B} (p^{[\mu\nu]}\partial_\mu A_\nu + p^{(\mu\nu)}\partial_\mu A_\nu + \rho^\nu \partial_\nu c - \frac{1}{2}p^{[\mu\nu]}p_{[\mu\nu]} - \frac{\xi}{2}p^{(\mu\nu)}p_{(\mu\nu)}) \, \mathrm{d}^d x =: \int_{\Gamma B} \mathcal{L}_{gf} \, \mathrm{d}^d x \quad (4.109)$$

where the integration is over a section ΓpB of pB, the pre-BRST super-multiphase space $\{(x^\mu, A_\alpha, p^{\alpha\mu}, c, \bar{c}^{\mu\nu}, b^{\mu\nu})\}$, or over a section ΓB of B, the BRST super-multiphase space $\{(x^\mu, A_\alpha, p^{\alpha\mu}, c, \rho^\mu)\}$, viewed as bundles over spacetime. This has DDW Hamiltonian:

$$\mathcal{H} = \frac{1}{2}p^{[\mu\nu]}p_{[\mu\nu]} + \frac{\xi}{2}p^{(\mu\nu)}p_{(\mu\nu)} \quad (4.110)$$

which is the correct result, expressed in the multiphase-space formalism can easily shown to result in the conventional soft-gauge-fixed Lagrangian density once the multimomenta are substituted for, using the equations of motion. The term $\rho^\nu\partial_\nu c$ in the Lagrangian above becomes zero, when the Euler-Lagrange equation for ρ^ν is employed. Substituting for the multimomenta, using the Euler-Lagrange equations, gives:

$$S_{BC}(\Gamma) \;=\; \int_\Gamma (\, \partial^{[\mu}A^{\nu]}\partial_\mu A_\nu + (2-\xi)\partial^{(\mu}A^{\nu)}\partial_\mu A_\nu \,) \, \mathrm{d}^d x \quad (4.111)$$

The Legendre transformation for c gives the canonical multimomenta for c, $\rho^\nu \approx \partial_\mu \bar{c}^{\mu\nu}$, which has been substituted in the last line of (4.109), and we can identify $b^{\mu\nu}$ and $p^{(\mu\nu)}$ which has also been substituted for in that last line.

If we set $\xi = 1$, then we see that the Lagrangian density can be written as $\partial^\mu A^\nu \partial_\mu A_\nu$ which is the same as obtained from the phase space gauge fixing in section 4.3.1 (compare (4.111) with (4.36)).

The Lagrangian density (4.109) is BRST invariant by construction (with $\delta_B \rho^\nu \approx \delta_B \partial_\mu \bar{c}^{\mu\nu} = \partial_\mu \bar{c}^{\mu\nu} = \partial_\mu b^{\mu\nu} = \partial_\mu p^{(\mu\nu)}$ which is zero on the constraint surface).

The BRST current is $T = J^\mu \mathrm{d}^{d-1}x_\mu \;= -c\;\partial_\alpha p^{(\alpha\mu)}\mathrm{d}^{d-1}x_\mu$ and

$$\delta_B \;\cdot = -\{-c\partial_\alpha p^{(\alpha\mu)}, \cdot\}_\mu \quad (4.112)$$

where the multi-bracket now is on a super-multiphase space with extra ghost canonical conjugate grassmannian coordinates c and ρ^α and therefore has an extra term $-\frac{\overleftarrow{\partial}}{\partial c} \wedge \frac{\overrightarrow{\partial}}{\partial \rho^\alpha}$. In the abelian case as here, the odd degrees of freedom decouple from the even as in conventional BRST.

The BRST variation on the BRST super-multiphase space $\mathcal{B} = \{(x, A_\alpha, p^{\alpha\mu}, c, \rho^\mu)\}$.

$$\begin{aligned} \delta_B A_\alpha &= \partial_\alpha c \\ \delta_B p^{\alpha\mu} &= 0 \\ \delta_B c &= 0 \\ \delta_B \rho^\mu &= \partial_\alpha p^{(\alpha\nu)} \end{aligned}$$

$$\delta_B x^\mu = 0 \tag{4.113}$$

The BRST variation is constructed from the differentials

$$\begin{aligned}
\mathrm{d}_1 g &= \frac{\partial}{\partial A_\nu}(g)\ \partial_\nu c \\
\mathrm{d}_1 c &= 0 \\
\mathrm{d}_1 \rho^\mu &= 0 \\
\mathrm{d}_1 x^\mu &= 0 \\
\delta g &= 0 \\
\delta c &= 0 \\
\delta \rho^\mu &= \partial_\alpha p^{(\alpha\mu)} \\
\delta x^\mu &= 0
\end{aligned} \tag{4.114}$$

The BRST variation of the Lagrangian density above is:

$$\delta_B \mathcal{L}_{gf} = \delta_B(p^{(\mu\nu)}\partial_\mu A_\nu + \rho^\nu \partial_\nu c) = (p^{(\mu\nu)}\partial_\mu \delta_B A_\nu + \delta_B \rho^\nu \partial_\nu c) = (p^{(\mu\nu)}\partial_\mu \partial_\nu c + \delta_B \partial_\alpha p^{(\alpha\nu)}\partial_\nu c) = 0 \tag{4.115}$$

where the last equality was achieved by integration by parts and ignoring the boundary term.

It is of interest to compare this with the well known result obtained from conventional phase-space BRST for the electromagnetic field in section 4.3.1. There the phase-space BRST charge was (in the following the summation over the x index signifies the integration over a spatial slice):

$$Q = -i(c_x \partial_i E^{ix} + \bar{\rho}_x p^{0x}) \tag{4.116}$$

and the phase-space BRST first order Lagrangian (4.34) was:

$$\begin{aligned}
L_{BP} &= \rho^x \partial_0 c_x + \bar{\rho}_x \partial_0 \bar{c}^x + p^{0x}\partial_0 A_{0x} + p^{ix}\partial_0 A_{ix} \\
&- \left(\frac{1}{2}(E^{ix}E_{ix} + B^{ix}B_{ix}) + \partial_i A_{0x} E^{ix} + \frac{\xi}{2} p^{0x} p_{0x} + p^{0x}\partial^i A_{ix} + \rho^x \bar{\rho}_x - \partial_i c_x \partial^i \bar{c}^x \right)
\end{aligned} \tag{4.117}$$

and the phase-space BRST Hamiltonian (4.35) was:

$$H_B = \frac{1}{2}(E^{ix}E_{ix} + B^{ix}B_{ix}) + \partial_i A_{0x} E^{ix} + \frac{\xi}{2} p^{0x} p_{0x} + p^{0x}\partial^i A_{ix} + \rho^x \bar{\rho}_x - \partial_i c_x \partial^i \bar{c}^x \tag{4.118}$$

It can be seen from these that the ghosts $(c, \bar{c})$ decouple from the grassmann even dynamical variables $(A_{ix}, A_{0x}, E_{ix}, p^{0x})$ and we achieve the correct form of the soft-gauge-fixed Hamiltonian for the abelian system.

4.7.2 Non-Abelian Yang-Mills

The configuration space action is

$$S[A^a_\mu(x)] = \int \frac{1}{2}|DA|^2 \mathrm{d}^d x = \int \frac{1}{2} D_{[\mu}A^a_{\nu]}\ D_{[\lambda}A^b_{\rho]}\ g^{\mu\lambda}g^{\nu\rho}\eta_{ab}\ \mathrm{d}^d x \tag{4.119}$$

, where A is the connection on a vector bundle over Minkowski spacetime with gauge group G with structure constants f^a_{bc} and Killing form η_{ab}. $D_\mu = \partial_\mu + g[A_\mu, \cdot] = \partial_\mu + g f^a_{bc} A^b_\mu$ is the covariant derivative. The action is invariant under variations $\delta_f A^a_\mu(x) = D_\mu f^a(x) = \partial_\mu f^a + g[A_\mu, f^a]$ for arbitrary sections f^a of the vector bundle. These are the gauge transformations and the infinitesimal gauge transformation algebra closes even off shell: $[\delta_{f_1}, \delta_{f_2}] = \delta_{f_3}$ where $f^a_3 = f^a_{bc} f^b_1 f^c_2$. The Legendre transformation maps to multimomenta $p^{\mu\nu}_a \approx\ D_{[\lambda}A^b_{\rho]}\ g^{\mu\lambda}g^{\nu\rho}\ \eta_{ba}\ =: \frac{1}{2}F^{\mu\nu}_a$ and primary constraints $p^{(\mu\nu)}_a \approx 0$. The DDW Hamiltonian is $H = p^{\mu\nu}_a \partial_\mu A^a_\nu\ - L = \frac{1}{2}p^{[\mu\nu]}_a p^a_{[\mu\nu]} - \frac{g}{2} p^{[\mu\nu]}_a f^a_{bc} A^b_\mu A^c_\nu + p^{(\mu\nu)}_a \partial_\mu A^a_\nu\ = \frac{1}{2}p^{[\mu\nu]}_a p^a_{[\mu\nu]} - \frac{g}{2} p^{[\mu\nu]}_a f^a_{bc} A^b_\mu A^c_\nu - \partial_\mu p^{(\mu\nu)}_a A^a_\nu$, employing integration by parts inside the first order action (4.122) for the last equality. The DeDonder Weyl equations of motion are:

$$\begin{aligned}
\partial_\mu A^a_\nu(x) - \frac{\partial \mathcal{H}}{\partial p^{\mu\nu}_a}(x, A^b_\kappa(x), p^{\lambda\kappa}_b(x)) = 0 \\
\partial_\mu p^{\mu\nu}_a(x) + \frac{\partial \mathcal{H}}{\partial A^a_\nu}(x, A^b_\kappa(x), p^{\lambda\kappa}_b(x)) = 0
\end{aligned} \tag{4.120}$$

which are, substituting for H,

$$\begin{aligned}
\partial_\mu A^a_\nu - p^a_{[\mu\nu]} + g f^a_{bc} A^b_\mu A^c_\nu - \partial_{(\mu}A^a_{\nu)} = 0 \\
\partial_\mu p^{\mu\nu}_a - \partial_\mu p^{(\mu\nu)}_a - g p^{[\mu\nu]}_d f^d_{ba} A^b = 0
\end{aligned} \tag{4.121}$$

which are also the Euler-Lagrange equations for a stationary point of the following first order action:

$$\begin{aligned}
S_{MP} = \int_{\Gamma J^1 \mathcal{E}^*} (p^{\mu\nu}_a \partial_\mu A^a_\nu - H)\mathrm{d}^d x = \int_{\Gamma J^1 \mathcal{E}^*} (p^{\mu\nu}_a \partial_\mu A^a_\nu - \frac{1}{2} p^{[\mu\nu]}_a p^a_{[\mu\nu]} + g f^a_{bc} A^b_\mu A^c_\nu - p^{(\mu\nu)}_a \partial_\mu A^a_\nu\ )\mathrm{d}^d x = \\
= \int_{\Gamma J^1 \mathcal{E}^*} (p^{[\mu\nu]}_a D_\mu A^a_\nu - \frac{1}{2} p^{[\mu\nu]}_a p^a_{[\mu\nu]})\mathrm{d}^d x
\end{aligned} \tag{4.122}$$

which has variation

$$\begin{aligned}
\delta S_{MP} = \delta \int_{\Gamma J^1 \mathcal{E}^*} (p^{\mu\nu}_a \partial_\mu A^a_\nu - H)\mathrm{d}^d x = \\
\int_{\Gamma J^1 \mathcal{E}^*} (\partial_\mu A^a_\nu - \frac{\partial \mathcal{H}}{\partial p^{\mu\nu}_a})\delta p^{\mu\nu}_a - (\partial_\mu p^{\mu\nu}_a + \frac{\partial \mathcal{H}}{\partial A^a_\nu})\delta A^a_\nu\ \mathrm{d}^d x\ +\ \int_{\partial \Gamma J^1 \mathcal{E}^*} [\,\delta A^a_\nu p^{\mu\nu}_a\,]\mathrm{d}S_\mu = \\
\int_{\Gamma J^1 \mathcal{E}^*} (\partial_\mu A^a_\nu - p^a_{[\mu\nu]} + g f^a_{bc} A^b_\mu A^c_\nu - \partial_{(\mu}A^a_{\nu)})\delta p^{\mu\nu}_a - (\partial_\mu p^{\mu\nu}_a - \partial_\mu p^{(\mu\nu)}_a - g p^{[\mu\nu]}_d f^d_{ba} A^b_\mu)\delta A^a_\nu\ \mathrm{d}^d x \\
+ \int_{\partial \Gamma J^1 \mathcal{E}^*} [\,\delta A^a_\nu p^{\mu\nu}_a\,]\mathrm{d}S_\mu
\end{aligned} \tag{4.123}$$

finally simplifying the DeDonder Weyl equations of motion, we obtain

$$D_{[\mu}A^a_{\nu]}(x) - p^a_{[\mu\nu]} = 0$$

$$D_\mu p_a^{[\mu\nu]} = 0 \tag{4.124}$$

By taking the covariant derivative D_λ of the first line above and antisymmetrizing, because $D_{[\lambda} D_\mu A^a_{\nu]} = 0$, we can eliminate the A and obtain equations of motion purely in terms of spacetime partial derivatives of the antisymmetric part of the multimomenta:

$$\begin{aligned} D_{[\lambda} p^a_{[\mu\nu]]} &= 0 \\ D_\mu p_a^{[\mu\nu]} &= 0 \end{aligned} \tag{4.125}$$

These are the generalization of Maxwell's equations, where $p^a_{[\mu\nu]} = \frac{1}{2}F^a_{\mu\nu}$, $p^a_{[0i]} = \frac{1}{2}F^a_{0i} = \frac{1}{2}E^a_i$ and $\epsilon_{ijk} p^a_{[jk]} = \frac{1}{2}\epsilon_{ijk} F^a_{jk} = B^a_i$ in the conventional notation for the gluon field.

From the primary constraints above,

$$p_a^{(\mu\nu)} = 0 \tag{4.126}$$

the constraints $D_\mu p_a^{(\mu\nu)} = 0$ generates the gauge transformations $\delta_f A^b_\rho = D_\rho f^a(x)$, $\delta_f p_b^{\mu\beta} = -g f^c f^a_{bc} p_a^{(\mu\beta)}$, under which the original Lagrangian (4.119) and the first order Lagrangian (4.122) are invariant, via the multi-bracket:

Variation of field configuration A:

$$-\delta_f A^b_\rho = \{f^a D_\mu p_a^{(\mu\alpha)}, A^b_\rho\}_\alpha = \mathrm{d}_V(f^a D_\mu p_a^{(\mu\alpha)}) \llcorner \Pi_\alpha \lrcorner \mathrm{d}_V(A^b_\rho) =$$

$$f^a(\partial_\mu p_a^{(\mu\alpha)} - g f^c_{ba} A^b p_c^{(\mu\alpha)}) \cdot (\frac{\overleftarrow{\partial}}{\partial A^c_\kappa} \wedge \frac{\overrightarrow{\partial}}{\partial p_c^{\kappa\alpha}}) \cdot A^b_\rho$$

$$= \partial_\mu f^a\ (p_a^{(\mu\alpha)} - g f^c_{ba} A^b p_c^{(\mu\alpha)}) \cdot (\frac{\overleftarrow{\partial}}{\partial A^c_\kappa} \wedge \frac{\overrightarrow{\partial}}{\partial p_c^{\kappa\alpha}}) \cdot A^b_\rho = -(\partial_\mu f^a\ - f^c g f^a_{bc} A^b) \left(\frac{\partial p_a^{(\mu\alpha)}}{\partial p_c^{\kappa\alpha}}\right) \left(\frac{\partial A^b_\rho}{\partial A^c_\kappa}\right)$$

$$= -D_\mu f^a \frac{1}{2}(\delta^\mu_\kappa \delta^\alpha_\alpha + \delta^\alpha_\kappa \delta^\mu_\alpha) \delta^c_a \delta^b_c \delta^\kappa_\rho = -D_\rho f^b \frac{1}{2}(d+1) \tag{4.127}$$

Variation of multimomenta p:

$$-\delta_f p_b^{\mu\beta} = \{f^a D_\mu p_a^{(\mu\alpha)}, p_b^{\mu\beta}\}_\alpha = -g f^c f^a_{bc} p_a^{(\mu\beta)} \tag{4.128}$$

Brackets of constraints with the DDW Hamiltonian:

$$\{H, p_a^{(\mu\alpha)}\}_\alpha = H \cdot (\frac{\overleftarrow{\partial}}{\partial A^c_\kappa} \wedge \frac{\overrightarrow{\partial}}{\partial p_c^{\kappa\alpha}}) \cdot p_a^{(\mu\alpha)} = -(\partial_\lambda p_a^{(\lambda\kappa)} - g p_c^{[\lambda\kappa]} f^c_{ba} A^b) \frac{1}{2}(\delta^\mu_\kappa \delta^\alpha_\alpha + \delta^\alpha_\kappa \delta^\mu_\alpha) = -D_\lambda p_a^{(\lambda\kappa)} \frac{1}{2}(d+1) \tag{4.129}$$

So $p_a^{(\mu\nu)} = 0$ are first class constraints.

We will follow the sequence for the electromagnetic field above to determine if it gives the correct result.

The example of the Yang-Mills field follows from the multiphase-space exposition in section 3.8. The construction below will be in two steps: initially we add a gauge fixing term to the the multiphase-space Lagrangian density, which now exists on an extended space of dynamical variables (the pre-super-multiphase space) $p\mathcal{B} = \{(x^\mu, A^a_\alpha, p^{\alpha\mu}_a, c^a, \bar{c}^{\mu\nu}_a, b^{\mu\nu})\}$ and in the second step we reduce to the super-multiphase space $\mathcal{B} = \{(x^\mu, A^a_\alpha, p^{\alpha\mu}_a, c^a, \rho^\mu_a)\}$.

To apply the BRST formalism to this model we replace the gauge variation parameter functions $f^a(x)$ by a grassmannian field $c^a(x)$ which we will adjoin to this model together with additional grassmanian field $\bar{c}^{\mu\nu}_a(x)$ and commuting field $b^{\mu\nu}_a(x)$ which are both symmetric in $\mu\nu$. The objective is to add a soft Lorentz gauge fixing term $b^{\mu\nu}_a \partial_\mu A^a_\nu - \frac{\xi}{2} b^{\mu\nu}_a b^a_{\mu\nu}$ to the first order Lagrangian density (4.122) in such a way that it remains invariant under the BRST variation δ_B on the pre-multiphase space $\{(x, A^a_\alpha, p^{\alpha\mu}_a, c^a, \bar{c}^{\mu\nu}_a, b^{\mu\nu})\}$, defined by

$$
\begin{aligned}
\delta_B A^a_\nu &= -D_\nu c^a \\
\delta_B p^{\mu\beta}_a &= g p^{\mu\beta}_b f^b_{ac} c^c \\
\delta_B c^a &= -\frac{1}{2}[c, c]^a = -\frac{1}{2} g f^a_{bc} c^b c^c \\
\delta_B \bar{c}^{\mu\nu}_a &= b^{\mu\nu}_a \\
\delta_B b^{\mu\nu}_a &= 0 \\
\delta_B x^\mu &= 0
\end{aligned}
\tag{4.130}
$$

With this variation it is clear that the first order Lagrangian density (4.122) remains invariant under this variation because we have simply replaced the gauge variation parameter $f(x)$ with $c(x)$. Adding a term of the form $\delta_B \Psi$, it will still be invariant because of the $\delta_B \delta_B = 0$ property of the variations above, which is due to the Jacobi identity on the structure constants: $f^a_{[bc} f^e_{d]a} = 0$ and the antisymmetry of $c^b c^c$. A suitable gauge fixing fermion is $\Psi = \bar{c}^{\mu\nu}_a (D_\mu A^a_\nu - \frac{\xi}{2} b^a_{\mu\nu}) = \bar{c}^{\mu\nu}_a (\partial_\mu A^a_\nu - \frac{\xi}{2} b^a_{\mu\nu})$, where ξ is a real number constant. This results in the additional Lorentz gauge fixing terms $\delta_B \Psi = b^{\mu\nu}_a \partial_\mu A^a_\nu - \frac{\xi}{2} b^{\mu\nu}_a b^a_{\mu\nu} + \bar{c}^{\mu\nu}_a \partial_\mu D_\nu c^a$, added to the Lagrangian density.

The BRST first order action is then:

$$
\begin{aligned}
S_\Psi = \int_{\Gamma pB} (p^{\mu\nu}_a \partial_\mu A^a_\nu - \mathcal{H} + \delta_B \Psi) \mathrm{d}^d x = \\
\int_{\Gamma pB} (p^{\mu\nu}_a \partial_\mu A^a_\nu - \frac{1}{2} p^{[\mu\nu]}_a p^a_{[\mu\nu]} - g p^{[\mu\nu]}_a f^a_{bc} A^b_\mu A^c_\nu - p^{(\mu\nu)}_a \partial_\mu A^a_\nu \\
+ (b^{\mu\nu}_a \partial_\mu A^a_\nu - \frac{\xi}{2} b^{\mu\nu}_a b^a_{\mu\nu} + \bar{c}^{\mu\nu}_a \partial_\mu D_\nu c^a)) \mathrm{d}^d x \\
= \int_{\Gamma B} (p^{[\mu\nu]}_a \partial_\mu A^a_\nu + p^{(\mu\nu)}_a \partial_\mu A^a_\nu + \rho^\nu_a \partial_\nu c^a \\
- \frac{1}{2} p^{[\mu\nu]}_a p^a_{[\mu\nu]} - \frac{\xi}{2} p^{(\mu\nu)}_a p^a_{(\mu\nu)} - g p^{[\mu\nu]}_a f^a_{bc} A^b_\mu A^c_\nu + g \rho^\nu_a f^a_{bc} A^b_\nu c^c) \mathrm{d}^d x
\end{aligned}
$$

$$= \int_{\Gamma B} (p_a^{[\mu\nu]} D_\mu A_\nu^a + p_a^{(\mu\nu)} \partial_\mu A_\nu^a + \rho_a^\nu D_\nu c^a - \frac{1}{2} p_a^{[\mu\nu]} p_{[\mu\nu]}^a - \frac{\xi}{2} p_a^{(\mu\nu)} p_{(\mu\nu)}^a)\ \mathrm{d}^d x$$

$$= \int_{\Gamma B} (p_a^{[\mu\nu]} D_\mu A_\nu^a + p_a^{(\mu\nu)} D_\mu A_\nu^a + \rho_a^\nu D_\nu c^a - \frac{1}{2} p_a^{[\mu\nu]} p_{[\mu\nu]}^a - \frac{\xi}{2} p_a^{(\mu\nu)} p_{(\mu\nu)}^a)\ \mathrm{d}^d x \tag{4.131}$$

where the integration is over a section ΓpB of pB, the pre-super-multiphase space , or over a section ΓB of B, the super-multiphase space , viewed as bundles over spacetime.

The Legendre transformation for c^a gives the canonical multimomenta for c^a, $\rho_a^\nu = -\partial_\mu \bar{c}_a^{\mu\nu}$, which has been substituted in the last line after integrating by parts and ignoring surface terms, and we can identify $b_a^{\mu\nu}$ and $p_a^{(\mu\nu)}$ which has also been substituted in the last line. The ρ_a^ν is the anti-field with opposite parity to A_ν^a and has pure ghost number 0, antifield number -1 and hence ghost number -1 [4].

If $\xi = 1$ this is

$$S_\Psi = \int_{\Gamma B} (p_a^{\mu\nu} D_\mu A_\nu^a + \rho_a^\nu D_\nu c^a - \frac{1}{2} p_a^{\mu\nu} p_{\mu\nu}^a)\ \mathrm{d}^d x$$

$$= \int_{\Gamma B} (p_a^{\mu\nu} \partial_\mu A_\nu^a + \rho_a^\nu \partial_\nu c^a + p_a^{\mu\nu} g f_{bc}^a A_\mu^b A_\nu^c + \rho_a^\nu g f_{bc}^a A_\nu^b c^c - \frac{1}{2} p_a^{\mu\nu} p_{\mu\nu}^a)\ \mathrm{d}^d x \tag{4.132}$$

So the gauge fixed DDW Hamiltonian for $\xi = 1$ is

$$\mathcal{H}_{gf} = \frac{1}{2} p_a^{\mu\nu} p_{\mu\nu}^a - p_a^{\mu\nu} g f_{bc}^a A_\mu^b A_\nu^c - \rho_a^\nu g f_{bc}^a A_\nu^b c^c \tag{4.133}$$

For general ξ it is

$$\mathcal{H}_{gf} = \frac{1}{2} p_a^{[\mu\nu]} p_{[\mu\nu]}^a + \frac{\xi}{2} p_a^{(\mu\nu)} p_{(\mu\nu)}^a + g p_a^{[\mu\nu]} f_{bc}^a A_\mu^b A_\nu^c - g \rho_a^\nu f_{bc}^a A_\nu^b c^c \tag{4.134}$$

In the functional integral of this Lagrangian in QFT, the gaussian integral

$$\int D[c] D[\rho] \exp\{-\frac{i}{\hbar} \int_{\Gamma B} (\rho_a^\nu \partial_\nu c^a + \rho_a^\nu g f_{bc}^a A_\nu^b c^c) d^d x\} \tag{4.135}$$

of the factor with the ghost terms shows that it is the same as integrating the ghost terms with the ρ_ν replaced by $-\partial_\nu c$. Thus the action integral can be written:

$$S_\Psi = \int_{\Gamma B} (p_a^{\mu\nu} D_\mu A_\nu^a + \partial^\nu c_a D_\nu c^a - \frac{1}{2} p_a^{\mu\nu} p_{\mu\nu}^a)\ \mathrm{d}^d x$$

$$= \int_{\Gamma B} (p_a^{\mu\nu} D_\mu A_\nu^a - c_a \partial^\nu D_\nu c^a - \frac{1}{2} p_a^{\mu\nu} p_{\mu\nu}^a)\ \mathrm{d}^d x \tag{4.136}$$

The result of the functional integral of the factor $\exp\{\frac{i}{\hbar} c_a \partial^\nu D_\nu c^a\}$ is the functional determinant $\det\{\partial^\nu D_\nu\}$, which is the well known result which can be arrived at by more conventional means.

This is BRST invariant by construction (with $\delta_B \rho_a^\nu = -\delta_B \partial_\mu \bar{c}_a^{\mu\nu} = -\partial_\mu b_a^{\mu\nu} = -\partial_\mu p_a^{(\mu\nu)}$). The BRST current is $T = J^\alpha \mathrm{d}^{d-1} x_\alpha = (c^a D_\mu p_a^{(\mu\alpha)} + \frac{1}{2} g f_{bc}^a \rho_a^\alpha c^b c^c) \mathrm{d}^{d-1} x_\alpha$ and $\delta_B \cdot = -\{J^\alpha, \cdot\}_\alpha$, where the multi-bracket now is on a super-multi-phase space with extra ghost canonical conjugate grassmannian coordinates c^a and ρ_a^α, and multibracket therefore has an extra term $-\frac{\overleftarrow{\partial}}{\partial c^a} \wedge \frac{\overrightarrow{\partial}}{\partial \rho_a^\alpha}$.

The Euler-Lagrange equation for ρ^ν_a is $D_\nu c^a \approx 0$, so $c(x)$ is covariantly constant relative to the connection $A(x)$ on solutions of the equations of motion.

The BRST variation δ_B on the BRST multiphase space $\mathcal{B} = \{(x^\mu, A^a_\alpha, p^{\alpha\mu}_a, c^a, \rho^\mu_a)\}$, is

$$\begin{aligned}
\delta_B A^a_\nu &= -D_\nu c^a \\
\delta_B p^{\alpha\mu}_a &= g p^{\alpha\mu}_b f^b_{ac} c^c \\
\delta_B c^a &= -\frac{1}{2}[c,c]^a = -\frac{1}{2} g f^a_{bc} c^b c^c \\
\delta_B \rho^\nu_a &= D_\alpha p^{(\alpha\nu)}_a - g\rho^\nu_b f^b_{ac} c^c = \partial_\alpha p^{(\alpha\nu)}_a + g p^{(\alpha\nu)}_b f^b_{ac} A^c_\alpha \; - g\rho^\nu_b f^b_{ac} c^c \\
\delta_B x^\mu &= 0
\end{aligned} \tag{4.137}$$

The BRST variation is constructed from the differentials

$$\begin{aligned}
\mathrm{d}_1 A^a_\nu &= -D_\nu c^a \\
\mathrm{d}_1 p^{\mu\beta}_a &= g p^{\mu\beta}_b f^b_{ac} c^c \\
\mathrm{d}_1 c^a &= -\frac{1}{2}[c,c]^a = -\frac{1}{2} g f^a_{bc} c^b c^c \\
\mathrm{d}_1 \rho^\alpha_a &= -g\rho^\alpha_b f^b_{ac} c^c \\
\mathrm{d}_1 x^\mu &= 0 \\
\delta g &= 0 \text{ where } g \text{ is a function on multiphase space} \\
\delta \rho^\nu_a &= D_\alpha p^{(\alpha\nu)}_a \\
\delta c^a &= 0 \\
\delta x^\mu &= 0
\end{aligned} \tag{4.138}$$

4.8 Summary

In this chapter, phase space BRST was reviewed and multiphase space BRST was developed with exaples of the electromagnetic field and Yang-Mills. The use of the Hamiltonian constraint and the hybrid technique was used to deal with a system with both primary and secondary constraints (electromagnetism). The multiphase space BRST structure will be used in the next two chapters on the sigma model.

Chapter 5

Multiphase-space BRST on a Riemannian target space

This construction will be for a field sigma model with diffeomorphism invariance in the target space, which is a Riemannian (or Lorentzian) manifold. The multiphase-space bundle is described in the section following together with the multi-Poisson bracket, and the BRST super-multiphase space, together with the super-multi-Poisson bracket, is described in the section after that. Then the special case when both the target space and the base space are almost hermitian manifolds is described in the last section. The latter is then employed in the next chapter, which analyses the Witten topological sigma model as a conjectured example of a multiphase-space BRST system constructed from a sigma model of J-holomorphic embeddings.

5.1 Sigma model

The configuration space fields are given locally by the coordinates u^i , $i = 1 \ldots n = \dim M$, on any coordinate patch on the Riemannian manifold (M, g), which is the target space. M is the fiber of the configuration bundle, $C = \Sigma \times M$, over the d-dimensional base space Σ, the latter having local coordinates σ^α , $\alpha = 0 \ldots d-1 = \dim \Sigma - 1$. The base space (Σ, h) is also a Riemannian or Lorentzian manifold. A field configuration $\phi = \phi(\sigma)$ is a section $\phi(\sigma) \in \Gamma(C)$ of the configuration bundle. This field configuration $\phi(\sigma)$ can be described in local coordinates $u^i(\sigma^\alpha)$, $i = 1 \ldots n = \dim M$, on a coordinate patch on the target manifold M and the base space Σ. The dynamics is given by the extremization of a local action:

$$\mathcal{S}[u^i(\sigma^\alpha)] \;=\; \int_\Sigma \mathrm{d}^d\sigma\, \mathcal{L}(u^i, \partial_\alpha u^i, \sigma^\alpha) \quad \text{or} \quad \mathcal{S}[\phi] \;=\; \int_\Sigma \mathrm{d}^d\sigma\, \mathcal{L}(j^1\phi) \tag{5.1}$$

where $\mathcal{L}(u^i, \partial_\alpha u^i, \sigma^\alpha)$ is the Lagrangian density, which, for a local action, is a function of σ^α, of the field configuration coordinates u^i, and its spacetime derivatives up to first order (higher order spacetime derivatives of the field will not be considered here). The Lagrangian can also be viewed as a function on the first jet bundle J^1C, which is the multivelocity phase space V, which has locally adapted coordinates $(u^i, v^i_\alpha, \sigma^\alpha)$ and the field configuration can be prolonged to a section of the first jet bundle $j^1\phi(\sigma^\alpha) = (u^i(\sigma^\alpha), \partial_\alpha u^i(\sigma^\alpha), \sigma^\alpha)$.

Sigma model multiphase space

The multimomenta p^α_i are the fiber coordinates of the dual first jet bundle. This is considered as a vector bundle over the configuration bundle C, and is the covariant multiphase space P, which has locally adapted coordinates σ^α , u^i, p^α_i. The canonical d form on P is $\Theta = du^i \wedge p^\alpha_i d\sigma_\alpha$ where $d\sigma_\alpha := \partial_\alpha \neg\, d\sigma^0 \wedge d\sigma^1 \wedge \ldots \wedge d\sigma^{d-1} = \partial_\alpha \neg\, d^d\sigma$ and the multisymplectic form is $\Omega = d\Theta = dp^\alpha_i \wedge du^i \wedge d\sigma_\alpha$.

Sign Convention: Note that Ω here is the negative of Ω used elsewhere in this paper.

The multi-Poisson bracket is defined (locally) as in previous chapters:

$$\begin{aligned}\{f, g\}_\alpha := f \overleftarrow{\mathrm{d}} \llcorner \Pi_\alpha \lrcorner \overrightarrow{\mathrm{d}} g = -\mathrm{d}f \llcorner \Pi_\alpha \lrcorner \mathrm{d}g = f \cdot\ \Pi_\alpha\ \cdot g := f \cdot (\frac{\overleftarrow{\partial}}{\partial u^i} \wedge \frac{\overrightarrow{\partial}}{\partial p^\alpha_i}) \cdot g \\ = \frac{\partial f}{\partial u^i}\frac{\partial g}{\partial p^\alpha_i} - \frac{\partial f}{\partial p^\alpha_i}\frac{\partial g}{\partial u^i}\end{aligned} \tag{5.2}$$

5.2 Sigma model super-multiphase space

From the configuration bundle C we construct the superconfiguration-space bundle SC over Σ with local coordinates $\sigma^\alpha, u^i, \eta^i$, where the additional odd parity (grassmannian) variables η^i are the fiber coordinates of the tangent bundle $\mathfrak{T}C$ of the configuration bundle C with fiber grassmann parity reversed. The super-multiphase space SP is constructed from SC in the same way that P is constructed from C and has locally adapted coordinates σ^α , $u^i, \eta^i, p^\alpha_i, \rho^\alpha_i$, where σ^α , u^i, p^α_i have even grassman parity and η^i, ρ^α_i have odd parity and ρ^α_i are the multimomenta dual to η^i. The canonical d form on SP is (from [9]):

$$\Theta = (\mathrm{d}u^i p^\alpha_i + D\eta^i \rho^\alpha_i) \wedge \mathrm{d}\sigma_\alpha \tag{5.3}$$

and the multisymplectic form is

$$\Omega = \mathrm{d}\Theta = (\mathrm{d}p^\alpha_i \wedge \mathrm{d}u^i - D\rho^\alpha_i \wedge D\eta^i - \frac{1}{2}R_{ijk}{}^l \eta^k \rho^\alpha_l \mathrm{d}u^i \wedge \mathrm{d}u^j\ ) \wedge \mathrm{d}\sigma_\alpha$$

$$= (\mathrm{d}p_i^\alpha \wedge \mathrm{d}u^i + \mathrm{d}\rho_i^\alpha \wedge \mathrm{d}\eta^i - \Gamma^l_{ki}\, \rho_l^\alpha \mathrm{d}\eta^i \wedge du^k + \Gamma^i_{kj}\, \eta^j\, \mathrm{d}\rho_i^\alpha \wedge \mathrm{d}u^k - \Gamma^l_{ik,j}\, \eta^k \rho_l^\alpha\, \mathrm{d}u^i \wedge \mathrm{d}u^j\,) \wedge \mathrm{d}\sigma_\alpha \quad (5.4)$$

Γ^i_{kj} is the Christoffel symbol and ${R_{ijk}}^l$ the Riemann tensor for the metric g on the target manifold M. The covariant derivative is employed because the η^i are the parity-reversed coordinates of the fibers of the tangent bundle TM.

Sign Convention: Note that Ω here is the negative of Ω used elsewhere in this paper.

The D is the covariant derivative $D\eta^j = \mathrm{d}\eta^j + \mathrm{d}u^i {\Gamma_{ik}}^j \eta^k$, $D\rho_j^\alpha = \mathrm{d}\rho_j^\alpha - \mathrm{d}u^i {\Gamma_{ij}}^k \rho_k^\alpha$ necessary because η^i transforms as a vector and ρ_j^α as a form, (unlike u^i which are scalar coordinate functions on M). This results in the Riemann tensor term in the multisymplectic form $\Omega = \mathrm{d}\Theta$.

The corresponding super-multi-Poisson bracket is :

$$\{f, g\}_\alpha := f \overleftarrow{\mathrm{d}} \llcorner \Pi_\alpha \lrcorner \overrightarrow{\mathrm{d}} g = -\mathrm{d}f \llcorner \Pi_\alpha \lrcorner \mathrm{d}g = f \cdot (\frac{\overleftarrow{\partial}}{\partial u^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha}) \cdot g \;-\; f \cdot (\frac{\overleftarrow{\partial}}{\partial \eta^i} \wedge \frac{\overrightarrow{\partial}}{\partial \rho_i^\alpha}) \cdot g \;+$$

$$\Gamma^j_{ik}\, \eta^k\; f \cdot (\frac{\overleftarrow{\partial}}{\partial \eta^j} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha}) \cdot g \;-\; \Gamma^j_{ik}\, \rho_j^\gamma\; f \cdot (\frac{\overleftarrow{\partial}}{\partial \rho_k^\alpha} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\gamma}) \cdot g \;-\; \frac{1}{2} {R_{ijk}}^l\, \rho_l^\gamma\, \eta^k\; f \cdot (\frac{\overleftarrow{\partial}}{\partial p_j^\gamma} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha}) \cdot g$$

$$= f \cdot (\frac{\overleftarrow{\partial}}{\partial u^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha}) \cdot g \;-\; f \cdot (\frac{\overleftarrow{D}}{D\eta^i} \wedge \frac{\overrightarrow{D}}{D\rho_i^\alpha}) \cdot g \;-\Gamma^l_{ik,j}\, \rho_l^\gamma\, \eta^k\; f \cdot (\frac{\overleftarrow{\partial}}{\partial p_j^\gamma} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha}) \cdot g \qquad (5.5)$$

where

$$\frac{\overrightarrow{D}}{D\rho_i^\alpha} := \frac{\overrightarrow{\partial}}{\partial \rho_i^\alpha} + \Gamma^i_{jk}\, \eta^k \frac{\overrightarrow{\partial}}{\partial p_j^\alpha} \quad \text{and} \quad \frac{\overleftarrow{D}}{D\eta^i} := \frac{\overleftarrow{\partial}}{\partial \eta^i} - \frac{\overleftarrow{\partial}}{\partial p_j^\alpha} \Gamma^k_{ij}\, \rho_k^\alpha \qquad (5.6)$$

The non-zero multi-Poisson brackets of the coordinate functions are:

$$\{u^j, p_i^\gamma\}_\alpha \;=\; \delta_i^j \delta_\alpha^\gamma \;=\; \{\eta^j, \rho_i^\gamma\}_\alpha \;=\; \{\rho_i^\gamma, \eta^j\}_\alpha \qquad (5.7)$$

$$\{p_j^\beta, p_i^\gamma\}_\alpha = \; {R_{ijk}}^l\; \eta^k\, \rho_l^{(\beta}\, \delta_\alpha^{\gamma)} \qquad (5.8)$$

$$\{\eta^j, p_i^\gamma\}_\alpha = \; {\Gamma_{ik}}^j\; \eta^k\, \delta_\alpha^\gamma \qquad (5.9)$$

$$\{\rho_j^\beta, p_i^\gamma\}_\alpha = -{\Gamma_{ij}}^k\; \rho_k^\gamma\, \delta_\alpha^\beta \qquad (5.10)$$

Writing $\bar{p}_i := p_i^\alpha \mathrm{d}^{d-1}\sigma_\alpha$, $\bar{\rho}_i := \rho_i^\alpha \mathrm{d}^{d-1}\sigma_\alpha$,

$$\{u^j, \bar{p}_i\} \;=\; \delta_i^j \;=\; \{\eta^j, \bar{\rho}_i\} \;=\; \{\bar{\rho}_i, \eta^j\} \qquad (5.11)$$

$$\{\bar{p}_j, \bar{p}_i\} = \; {R_{ijk}}^l\; \eta^k\, \bar{\rho}_l \qquad (5.12)$$

$$\{\eta^j, \bar{p}_i\} = \; {\Gamma_{ik}}^j\; \eta^k \qquad (5.13)$$

$$\{\bar{\rho}_j, \bar{p}_i\} = -{\Gamma_{ij}}^k\; \bar{\rho}_k \qquad (5.14)$$

Other multi-Poisson brackets of coordinate functions being zero.

$$X_{u^i} \lrcorner \Omega = \mathrm{d}u^i \qquad (5.15)$$

$$\{u^i, \cdot\} = \{u^i, \cdot\}_\alpha\, \partial^\alpha = \frac{\partial}{\partial p_i^\alpha} \wedge \partial^\alpha = X_{u^i} \tag{5.16}$$

Similarly, for $\bar{p}_i = p_i^\alpha \mathrm{d}\sigma_\alpha$:

$$X_{\bar{p}_i} = \{\bar{p}_i, \cdot\} = \{p_i^\alpha \mathrm{d}\sigma_\alpha \llcorner\, , \cdot\}_\beta\, \partial^\beta = -\frac{\partial}{\partial u^i} - \Gamma^j_{ik}\, \eta^k \frac{\partial}{\partial \eta^j} \;+\; \Gamma^j_{ik}\, \rho^\gamma_j \frac{\partial}{\partial \rho^\gamma_k} \;+\; R_{ijk}{}^l\, \rho^\gamma_l\, \eta^k \frac{\partial}{\partial p^\gamma_j} \tag{5.17}$$

then $X_{\bar{p}_i} \lrcorner\, \Omega = \mathrm{d}\bar{p}_i = \mathrm{d}p_i^\alpha \wedge \mathrm{d}\sigma_\alpha$

Also:

$$X_{\bar{\rho}_i} = \{\bar{\rho}_i, \cdot\} = \{\rho_i^\alpha \mathrm{d}\sigma_\alpha \llcorner\, , \cdot\}_\beta\, \partial^\beta = \frac{\partial}{\partial \eta^i} \;-\; \Gamma^k_{ij}\, \rho^\alpha_k \frac{\partial}{\partial p^\alpha_j} \;=:\; \frac{D}{D\eta^i} \tag{5.18}$$

then $X_{\bar{\rho}_i} \lrcorner\, \Omega = \mathrm{d}\bar{\rho}_i = \mathrm{d}\rho_i^\alpha \wedge \mathrm{d}\sigma_\alpha$ and $\Omega \llcorner X_{\bar{\rho}_i} = -(-1)^{d(d+1)/2}\mathrm{d}\bar{\rho}_i = \mathrm{d}\bar{\rho}_i$ if $d = 1$ or 2.

$$X_{\eta^i} = \{\eta^i, \cdot\} = \{\eta^i, \cdot\}_\beta\, \partial^\beta = \left(\frac{\partial}{\partial \rho^\alpha_i} + \Gamma^i_{kj}\, \eta^k \frac{\partial}{\partial p^\alpha_j}\right) \wedge \partial^\alpha =: \frac{D}{D\rho^\alpha_i} \wedge \partial^\alpha \;\; =: \;\; \frac{\bar{D}}{D\bar{\rho}_i} \tag{5.19}$$

then $X_{\eta^i} \lrcorner\, \Omega = \mathrm{d}\eta^i$ and $\Omega \llcorner X_{\eta^i} = -(-1)^{d(d+1)/2}\mathrm{d}\eta^i = \mathrm{d}\eta^i$ if $d = 1$ or 2.

5.3 Almost-Hermitian manifolds

We will be considering an almost complex structure, a (1,1)-tensor field J such that $J^2 = -1$ on the target space M, in addition to the metric in the previous section. When these two structures are compatible, $g(JX, JY) = g(X, Y)$, this defines an almost Hermitian manifold. There is an appendix (appendix A) which summarizes some relevant definitions and identities involving various categories of almost Hermitian manifolds, such as Hermitian, Kähler, almost Kähler, and quasi-Kähler.

J-holomorphic curves

The field configurations are sections of the bundle $\Sigma \times M$ with fiber M over a base space Σ. We will consider in the following and the next chapter spaces M , Σ, with certain properties and maps between them also with specified properties.

The base space is an almost Hermitian manifold (Σ, h, ε) with metric h and compatible complex structure ε and the target space is a d-dimensional almost Hermitian manifold (M, g, J) with almost complex structure J and compatible metric g. Local coordinates on Σ are denoted by $(\sigma^\alpha\ ;\ \alpha = 0, \ldots, d-1)$ and on M by $(u^i\ ;\ i = 1 \ldots N)$. A J-holomorphic (also called pseudo-holomorphic) curve is a mapping of a Riemann surface to an almost complex manifold which preserves the almost complex structure, as explained in the next two paragraphs.

We can define a pair of projectors $\overset{+}{\Pi}{}^{i\beta}_{\alpha j} = \frac{1}{2}(\delta^{\beta}_{\alpha}\delta^{i}_{j} + \varepsilon_{\alpha}{}^{\beta} J^{i}{}_{j})$ and $\overset{-}{\Pi}{}^{i\beta}_{\alpha j} = \frac{1}{2}(\delta^{\beta}_{\alpha}\delta^{i}_{j} - \varepsilon_{\alpha}{}^{\beta} J^{i}{}_{j})$ which satisfy the projector properties $\overset{-}{\Pi}{}^{\gamma i}_{k\alpha}\overset{+}{\Pi}{}^{\alpha\beta}_{ij} = 0$ and $\overset{-}{\Pi}{}^{\gamma i}_{k\alpha}\overset{-}{\Pi}{}^{\alpha j}_{i\beta} = \overset{-}{\Pi}{}^{\gamma j}_{k\beta}$ and $\overset{+}{\Pi}{}^{\gamma i}_{k\alpha}\overset{+}{\Pi}{}^{\alpha j}_{i\beta} = \overset{+}{\Pi}{}^{\gamma j}_{k\beta}$. Even though defined locally, these are well defined tensors on all of $\Sigma \times M$, because J, ε, δ are tensors defined on the manifolds. The projectors which map a tensor $p = p^{\alpha}_{i}\mathrm{d}\sigma_{\alpha} \wedge \mathrm{d}u^{i}$ to tensors $\overset{\pm}{p} = \overset{\pm}{p}{}^{\alpha}_{i}\mathrm{d}\sigma_{\alpha} \wedge \mathrm{d}u^{i}$, where $p = \overset{+}{p} + \overset{-}{p}$:

$$\overset{+}{p}{}^{\alpha}_{i} := \overset{+}{\Pi}{}^{\alpha j}_{i\beta}\, p^{\beta}_{j} := \frac{1}{2}(\delta^{\alpha}_{\beta}\delta^{j}_{i} + \varepsilon^{\alpha}{}_{\beta} J_{i}{}^{j})p^{\beta}_{j} \quad \text{and} \quad \overset{-}{p}{}^{\alpha}_{i} := \overset{-}{\Pi}{}^{\alpha j}_{i\beta}\, p^{\beta}_{j} := \frac{1}{2}(\delta^{\alpha}_{\beta}\delta^{j}_{i} - \varepsilon^{\alpha}{}_{\beta} J_{i}{}^{j})p^{\beta}_{j} \tag{5.20}$$

thus $\overset{-}{p}{}^{\alpha}_{i}$ is defined to be negative projected (or pseudo-holomorphic) part of p^{α}_{i}, and $\overset{+}{p}{}^{\alpha}_{i}$ is defined to be positive projected (or pseudo-antiholomorphic) part of p^{α}_{i}, with the projectors constructed from the almost complex structure tensors of Σ and M.

Let ϕ be a section of the bundle $\Sigma \times M$, in local coordinates $u^{i}(\sigma^{\alpha})$. Let $p^{i}_{\alpha} := \partial_{\alpha}u^{i}$ and $p^{\alpha}_{i} := g_{ij}h^{\alpha\beta}p^{j}_{\beta}$. If $\overset{+}{p}{}^{\alpha}_{i} = 0$ througout a section ϕ, (this means that $p^{\alpha}_{\bar{i}} := J_{i}{}^{j}p^{\alpha}_{j} = \varepsilon^{\alpha}{}_{\beta}p^{\beta}_{i} =: p^{\bar{\alpha}}_{i}$): this defines an almost holomorphic section and $p^{\alpha}_{\bar{j}} = \bar{p}^{\alpha}_{\bar{j}}$. If (Σ, h, ε) is a Hermitian manifold, that is, ε is a an integrable almost complex structure and so a complex structure, with h a compatible metric, then the above section ϕ is a J-holomorphic map. If also J in (M, J, g) is a an integrable complex structure and so a complex structure, with g a compatible metric, then the above section ϕ is a holomorphic map $\phi : \Sigma \longrightarrow M$.

The particular case which we will be concerned with, the base space is two dimensional ($d = 2$), and a two dimensional almost Hermitian manifold is necessarily a Riemann surface.

Supermultiphase-space BRST for J-holomorphic curves on an almost Hermitian manifold target space

We define constraints $T_{i'} = \mathrm{d}\sigma_{\gamma'}\ \Pi^{\gamma' i}_{i'\gamma}\ p^{\gamma}_{i}$ for one or the other of the projectors, then the multivector multi-Poisson bracket acting on the constraint

$$T_{\epsilon} := \epsilon^{i'}(u^{i}, \sigma^{\alpha})T_{i'} = \epsilon^{i'}(u^{i}, \sigma^{\alpha})\ \mathrm{d}\sigma_{\gamma'}\ \Pi^{\gamma' i}_{i'\gamma}\ p^{\gamma}_{i} \tag{5.21}$$

gives the generalized Hamiltonian vector field of that constraint:

$$X_{\epsilon^{i'}T_{i'}} = \{\epsilon^{i'}T_{i'}, \cdot\} = \{\,\epsilon^{i'}\mathrm{d}\sigma_{\gamma'}\ \Pi^{\gamma' i}_{i'\gamma}\ p^{\gamma}_{i}, \cdot\} = (\epsilon^{i'}T_{i'}) \cdot\ \Pi \cdot :=$$

$$(\epsilon^{i'}T_{i'}) \cdot (\frac{\overleftarrow{\partial}}{\partial u^{i}} \wedge \frac{\partial}{\partial \sigma_{\alpha}} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\alpha}_{i}}) \cdot\ = -\mathrm{d}_{v}(\epsilon^{i'}T_{i'}) \llcorner (\frac{\overleftarrow{\partial}}{\partial u^{i}} \wedge \frac{\partial}{\partial \sigma_{\alpha}} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\alpha}_{i}}) \cdot$$

$$= -\mathrm{d}_{v}(\epsilon^{i'}\ \mathrm{d}\sigma_{\gamma'}\ \Pi^{\gamma' i}_{i'\gamma}\ p^{\gamma}_{i}) \llcorner (\frac{\overleftarrow{\partial}}{\partial u^{i}} \wedge \frac{\partial}{\partial \sigma_{\alpha}} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\alpha}_{i}}) \cdot$$

$$= -(\epsilon^{i'}\ \Pi^{\gamma' i}_{i'\gamma}\ \mathrm{d}p^{\gamma}_{i} \wedge \mathrm{d}\sigma_{\gamma'} + \epsilon^{i'}\ (\Pi^{\gamma' i}_{i'\gamma})_{,k}\, p^{\gamma}_{i}\mathrm{d}u^{k} \wedge \mathrm{d}\sigma_{\gamma'} + \epsilon^{i'}_{,k}\ \Pi^{\gamma' i}_{i'\gamma}\ p^{\gamma}_{i}\ \mathrm{d}u^{k} \wedge \mathrm{d}\sigma_{\gamma'}) \llcorner (\frac{\overleftarrow{\partial}}{\partial u^{i}} \wedge \frac{\partial}{\partial \sigma_{\alpha}} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\alpha}_{i}})$$

$$= -\mathrm{d}\sigma_{\gamma'}\ \epsilon^{i'}\ \Pi^{\gamma' i}_{i'\gamma}\ \frac{\partial}{\partial\sigma_\gamma}\wedge\frac{\partial}{\partial u^i}\ +\ \epsilon^{i'}\ (\Pi^{\gamma' i}_{i'\gamma})_{,k}\ p^\gamma_i\ \frac{\partial}{\partial p^{\gamma'}_k}\ +\ \epsilon^{i'}_{,k}\ \Pi^{\gamma' i}_{i'\gamma}\ p^\gamma_i\ \frac{\partial}{\partial p^{\gamma'}_k} \tag{5.22}$$

We check that $X_{\epsilon^{i'}T_{i'}} \lrcorner\, \Omega = \mathrm{d}_v(\epsilon^{i'}T_{i'})$. The left hand side is

$$(\ -\mathrm{d}\sigma_{\gamma'}\ \epsilon^{i'}\ \Pi^{\gamma' i}_{i'\gamma}\ \frac{\partial}{\partial\sigma_\gamma}\wedge\frac{\partial}{\partial u^i}\ +\ \epsilon^{i'}\ (\Pi^{\gamma' i}_{i'\gamma})_{,k}\ p^\gamma_i\ \frac{\partial}{\partial p^{\gamma'}_k}\ +\ \epsilon^{i'}_{,k}\ \Pi^{\gamma' i}_{i'\gamma}\ p^\gamma_i\ \frac{\partial}{\partial p^{\gamma'}_k})\ \lrcorner\ \mathrm{d}p^\alpha_i\wedge\mathrm{d}u^i\wedge\mathrm{d}\sigma_\alpha$$

$$= \epsilon^{i'}\ \Pi^{\gamma' i}_{i'\gamma}\ \mathrm{d}p^\gamma_i\wedge\mathrm{d}\sigma_{\gamma'} +\ \epsilon^{i'}\ (\Pi^{\gamma' i}_{i'\gamma})_{,k}\ p^\gamma_i \mathrm{d}u^k\wedge\mathrm{d}\sigma_{\gamma'} +\ \epsilon^{i'}_{,k}\ \Pi^{\gamma' i}_{i'\gamma}\ p^\gamma_i\ \mathrm{d}u^k\wedge\mathrm{d}\sigma_{\gamma'} \tag{5.23}$$

The right hand side is

$$\mathrm{d}_v(\epsilon^{i'}T_{i'}) = \epsilon^{i'}\ \Pi^{\gamma' i}_{i'\gamma}\ \mathrm{d}p^\gamma_i\wedge\mathrm{d}\sigma_{\gamma'} +\ \epsilon^{i'}\ (\Pi^{\gamma' i}_{i'\gamma})_{,k}\ p^\gamma_i \mathrm{d}u^k\wedge\mathrm{d}\sigma_{\gamma'} +\ \epsilon^{i'}_{,k}\ \Pi^{\gamma' i}_{i'\gamma}\ p^\gamma_i\ \mathrm{d}u^k\wedge\mathrm{d}\sigma_{\gamma'} \tag{5.24}$$

We want to calculate the multi-Poisson bracket of two constraints:

$$\{T_\epsilon, T_\eta\} = \{\epsilon^{i'}T_{i'}, \eta^{i'}T_{i'}\} = X_{\epsilon^{i'}T_{i'}}\lrcorner\ \mathrm{d}_v(\eta^{i'}T_{i'})\ =$$

$$(-\mathrm{d}\sigma_{\gamma'}\ \epsilon^{i'}\ \Pi^{\gamma' i}_{i'\gamma}\ \frac{\partial}{\partial\sigma_\gamma}\wedge\frac{\partial}{\partial u^i} + (\epsilon^{i'}\ \Pi^{\gamma' i}_{i'\gamma})_{,k}\ p^\gamma_i\ \frac{\partial}{\partial p^{\gamma'}_k}\ )\ \lrcorner\ (\ \eta^{i'}\ \Pi^{\gamma' i}_{i'\gamma}\ \mathrm{d}p^\gamma_i + (\eta^{i'}\ \Pi^{\gamma' i}_{i'\gamma})_{,k}\ p^\gamma_i\ \mathrm{d}u^k\ )\wedge\mathrm{d}\sigma_{\gamma'}$$

$$= \mathrm{d}\sigma_{\gamma'}\ (\ -(-1)\ \epsilon^{i'}\ \Pi^{\gamma' i}_{i'\gamma}\ (\eta^{j'}\Pi^{\gamma j}_{j'\delta})_{,i}\ p^\delta_j\ +\ \eta^{i'}\ \Pi^{\gamma' i}_{i'\gamma}\ (\epsilon^{j'}\Pi^{\gamma j}_{j'\delta})_{,i}\ p^\delta_j\ )$$

$$= -\mathrm{d}\sigma_{\gamma'}\ (\ \epsilon^{i'}\eta^{j'}_{,i}\ \Pi^{\gamma' i}_{i'\gamma}\ \Pi^{\gamma j}_{j'\delta}\ p^\delta_j\ +\ \epsilon^{i'}\eta^{j'}\ \Pi^{\gamma' i}_{i'\gamma}\ \Pi^{\gamma j}_{j'\delta,i}\ p^\delta_j\ - (\epsilon\leftrightarrow\eta)\ ) \tag{5.25}$$

Conjecture

We conjecture that, under suitable conditions, one would obtain the following first class bracket algebra of the constraints:

$$\{T_\epsilon, T_\eta\}\ \approx\ \pm\ \mathrm{d}\sigma_{\gamma'}\ \epsilon^{i'}\eta^{j'}(J_{[i'}{}^m D_{j']}J_m{}^l)\ \Pi^{\gamma' j}_{l\ \gamma}\ p^\gamma_j\ =:\ \pm\ \epsilon^{i'}\eta^{j'}C^l_{i'j'}\ T_l \tag{5.26}$$

The super-multi-Poisson bracket of the projected multimomenta (in super-multiphase space) are:

$$\{{}^+p^{\beta'}_{j'}, {}^+p^{\gamma'}_{i'}\}_\alpha = \{{}^+\ \Pi^{\beta' j}_{j'\beta}\ p^\beta_j\ ,\ {}^+\ \Pi^{\gamma' i}_{i'\gamma}\ p^\gamma_i\}_\alpha =$$

$$\frac{1}{2}\big(R_{i'j'k}{}^l\ \eta^k\ \rho^{[\beta'}_l\ \delta^{\gamma']}_\alpha + J_{i'}{}^i J_{j'}{}^j R_{ijk}{}^l\ \eta^k\ \rho^\kappa_l\ \epsilon^{[\beta'}{}_\kappa\ \epsilon^{\gamma']}{}_\alpha + J_{i'}{}^i R_{ij'k}{}^l\ \eta^k\ \rho^\kappa_l\ \delta^{[\beta'}_\kappa\ \epsilon^{\gamma']}{}_\alpha + J_{j'}{}^j R_{i'jk}{}^l\ \eta^k\ \rho^\kappa_l\ \delta^{[\gamma'}_\alpha\ \epsilon^{\beta']}{}_\kappa\big)$$

$$+\frac{1}{2}\big(D_{[i'}\ J_{j']}{}^j\ \delta^{[\gamma'}{}_\alpha + J_{[i'}{}^k D_k\ J_{j']}{}^j\ \epsilon^{[\gamma'}{}_\alpha\big)\ \epsilon^{\beta']}{}_\kappa\ p^\kappa_j \tag{5.27}$$

when β' and α are contracted, the above becomes:

$$\{{}^+p^\alpha_{j'}, {}^+p^{\gamma'}_{i'}\}_\alpha = \{{}^+\ \Pi^{\alpha j}_{j'\beta}\ p^\beta_j\ ,\ {}^+\ \Pi^{\gamma' i}_{i'\gamma}\ p^\gamma_i\}_\alpha =$$

$$\frac{1}{2}\big(R_{i'j'k}{}^l\ \eta^k\ \rho^{\gamma'}_l\ (1-d) +\ J_{i'}{}^i J_{j'}{}^j R_{ijk}{}^l\ \eta^k\ (-\rho^{\gamma'}_l\ + 0\ \rho^\kappa_l\ \epsilon^{\gamma'}{}_\kappa)\ +\ J_{i'}{}^i R_{ij'k}{}^l\ \eta^k\ \rho^\kappa_l\ \epsilon^\gamma{}_\kappa(1-d)$$

$$
\begin{aligned}
&+ J_{j'}{}^{j} R_{i'jk}{}^{l}\, \eta^k \rho_l^\kappa \epsilon^\gamma{}_\kappa (1-0)) \\
&- C_{i'j'}{}^{l}\, {}^{+}p_l^{\gamma'} + N_{i'j'}{}^{l}\, p_l^{\gamma'}
\end{aligned}
\tag{5.28}
$$

where $N_{i'j'}{}^{l}$ is the Nijenhuis tensor, $C^l_{i'j'} := (J_{[i'}{}^{m} D_{j']} J_m{}^{l})$, and the quasi-Kähler condition $J_i{}^{i'}\, D_{i'}\, J_j{}^{k} = J_j{}^{j'}\, D_i\, J_{j'}{}^{k}$ on (M, J, g) is imposed. This is antisymmetric in the $[i'j']$ indices except for the terms with a single J factor contracting with R. Note that

$$
- C_{i'j'}{}^{l}\, {}^{+}p_l^{\gamma'} + N_{i'j'}{}^{l}\, p_l^{\gamma'} = - C_{i'j'}{}^{l}\, {}^{-}p_l^{\gamma'}
\tag{5.29}
$$

When d=2 the bracket simplifies to:

$$
\begin{aligned}
&\{{}^{+}p^\alpha_{j'}, {}^{+}p^{\gamma'}_{i'}\}_\alpha = \{{}^{+}\,\Pi^{\alpha\, j}_{j'\beta}\; p^\beta_j\,,\, {}^{+}\,\Pi^{\gamma' i}_{i'\gamma}\; p^\gamma_i\}_\alpha = \\
-\frac{1}{2}\big(&R_{i'j'k}{}^{l}\, \eta^k \rho_l^{\gamma'} + J_{i'}{}^{i} J_{j'}{}^{j} R_{ijk}{}^{l}\, \eta^k \rho_l^{\gamma'} - 2\, J_{(i'|}{}^{i} R_{i|j')k}{}^{l}\, \eta^k \rho_l^\kappa \epsilon^\gamma{}_\kappa) \\
&- C_{i'j'}{}^{l}\, {}^{+}p_l^{\gamma'} + N_{i'j'}{}^{l}\, p_l^{\gamma'}
\end{aligned}
\tag{5.30}
$$

The bracket of a projected multimomenta with multimomenta are:

$$
\begin{aligned}
&\{{}^{+}p^{\beta'}_{j'}\,,\, p^{\gamma'}_{i'}\}_\alpha = \{{}^{+}\,\Pi^{\beta' j}_{j'\beta}\; p^\beta_j\,,\, p^{\gamma'}_{i'}\}_\alpha = \\
\frac{1}{2}\big(&R_{i'j'k}{}^{l}\, \eta^k \rho_l^{[\beta'}\, \delta_\alpha^{\gamma']} + J_{i'}{}^{i} R_{ij'k}{}^{l}\, \eta^k \rho_l^\kappa\, \delta_\kappa^{[\beta'}\, \epsilon^{\gamma']}{}_\alpha + D_{[i'}\, J_{j']}{}^{j}\, \delta^{[\gamma'}{}_\alpha\, \epsilon^{\beta']}{}_\kappa\, p_j^\kappa\big)
\end{aligned}
\tag{5.31}
$$

when β' and α are contracted, the above becomes:

$$
\{{}^{+}p^\alpha_{j'}, p^{\gamma'}_{i'}\}_\alpha = \frac{1}{2}\big(R_{i'j'k}{}^{l}\, \eta^k \rho_l^{\gamma'}\, (1-d) + J_{j'}{}^{j} R_{i'jk}{}^{l}\, \eta^k \rho_l^\kappa \epsilon^{\gamma'}{}_\kappa (1-0) + D_{i'}\, J_{j'}{}^{j}\, \epsilon^{\beta'}{}_\kappa\, p_j^\kappa\,\big)
\tag{5.32}
$$

When d=2 the bracket simplifies to:

$$
\{{}^{+}p^\alpha_{j'}, p^{\gamma'}_{i'}\}_\alpha = \frac{1}{2}\big(-R_{i'j'k}{}^{l}\, \eta^k \rho_l^{\gamma'} + J_{j'}{}^{j} R_{i'jk}{}^{l}\, \eta^k \rho_l^\kappa \epsilon^{\gamma'}{}_\kappa + D_{i'}\, J_{j'}{}^{j}\, \epsilon^{\beta'}{}_\kappa\, p_j^\kappa\,\big)
\tag{5.33}
$$

This is antisymmetric in the $[i'j']$ indices except for the term with a single J factor contracting with R, which is symmetric in those indices and will be cancelled if these indices are contracted with an antisymmetric tensor, such as $\eta^{i'}\eta^{j'}$.

5.4 Summary

In this chapter, the multiphase-space BRST was developed for the sigma model and then for a Riemann surface base space and an almost Hermitian target space. The corresponding super multi-Poisson brackets were calculated J-antiholomorphic projected multimomenta. The results for J-holomorphic projected multimomenta are similar.

The above brackets will be seen in the next chapter to resemble part of the BRST-like gauge fixing term in Witten model and it will be conjectured that the gauge fixing term is the result of the multibracket between a mutiphase space BRST current and a gauge fixing fermion.

Chapter 6

The topological sigma model

In this chapter a multiphase-space BRST technique is applied to a sigma model with gauge symmetries, the sigma model of J-holomorphic embeddings. The resullt is compared to Witten's topological sigma model whose the relevant part is described near the end of the chapter in section 6.4.

The super-phase space structure described in the previous chapter is applied to the dynamics of a sigma model of J-holomorphic embeddings, which means a Riemann surface which is embedded in an almost hermitian manifold (i.e. with an almost complex structure and a compatible metric). Our starting point is a quadratic Lagrangian density (6.3) which depends on the perturbation from the J-holomorphicity (also called pseudoholomorphicity) of the embedding of the Riemann surface (the base space) into the almost hermitian manifold (the target space), in that the Lagrangian density is a kinetic term $\partial_\alpha u^i \overset{+}{\Pi}{}_{ij}^{\alpha\beta} \partial_\beta u^j$ which measures the deviation of the embedding from J-holomorphicity. Thus the Lagrangian density is such that the integral over the Riemann surface is independent of the coordinatization of the Riemann surface and the almost hermitian manifold in which it is embedded. The action is minimized by any J-holomorphic embedding. There is thus a gauge symmetry of perturbations which preserve the condition of J-holomorphicity. The minimization picks out the J-holomorphic diffeomorphism classes of J-holomorphic curves in the almost hermitian manifold M, if one were to mod out the J-holomorphic diffeomorphisms. This is a topological invariant. The multiphase space is constructed via the multi-Legendre transformation which leads to the primary constraint that the J-holomorphic part of the multimomenta are constrained to be zero.

The objective is to have these generate the gauge variations via the multi-Poisson bracket of the previous chapter and we would expect the constraint algebra to be that of a first class multiphase-space constraint algebra.

The gauge symmetry of J-holomorphic variations might be dealt with by employing the multiphase-space BRST technique developed in chapter 4, leading to an enlargement of the multiphase space to a super-multiphase space where the gauge variation parameters are promoted to odd fields with corresponding odd multimomenta. The original local gauge symmetry of the Lagrangian then becomes a part of a global BRST symmetry of the Lagrangian in the super-multiphase space shown in the previous chapter. The BRST charge J^{α} (in the multiphase-space setting a $d-1$-form), which generates the BRST variation δ_B, would be constructed starting from the multiphase-space generators of the gauge symmetry in such a way that the variation is nilpotent, $\delta_B^2 = 0$. This would done employing the algebra of the multiphase-space generators as shown in chapter 4. This would allow a a gauge fixing term of the form $\delta_B \Psi$ to be added to the original Lagrangian without losing the BRST symmetry because of the nilpotence property $\delta_B^2 \Psi = 0$

In section (6.4) the Witten topological sigma model [30] is briefly summarized and compared to the multiphase-space BRST construction. In that section the relevant aspects of the Witten model are reproduced to show the BRST-like features of his construction (which he remarked upon in his original paper) and the role of some of his fields as multimomenta (first noted by Hrabak [84]).

The calculations are done to attempt to construct a multiphase-space BRST model which reproduces the Witten model. In particular, we want to check whether the gauge variation in the multiphase space topological sigma model is the same as the grassmann even part of Witten's BRST-like variation. We also seek the conditions under which the algebra of the gauge variation generators close under the multi-Poisson brackets.

6.1 Multiphase-space topological sigma model

The model is a sigma model in which the base space is a Riemann surface Σ with hermitian metric h, compatible complex structure ε, and the target space is a d-dimensional almost hermitian manifold M with almost complex structure J and compatible metric g. The target manifold M is such that it has an almost complex structure J, which imposes limitations on possible topologies, but nevertheless any almost complex manifold can be given a compatible metric to produce an almost hermitian structure on the manifold. Because the model is about finding global topological invariants, the restriction on metrics is not an obstacle.

Local coordinates on the base space (Riemann surface) Σ are denoted by $(\sigma^{\alpha}\ ;\ \alpha = 0, 1)$ and on the target space M by $(u^i\ ;\ i = 1 \ldots d)$.

We first define the fields $u^i(\sigma^\alpha)$, where $u^i, i = 1 \dots \dim M$ are local coordinates on the target manifold M. The base space, with local coordinates $\alpha = 0,1$, is the Riemann surface Σ. A field configuration is a specific map $\Sigma \longrightarrow M$ and corresponds to an embedding of a string with worldsurface Σ into an almost hermitian manifold M. In multiphase space a field configuration is a section $(u^i(\sigma^\alpha), p_i^\alpha(\sigma^\alpha))$ over the base space Σ. The extra multimomenta fields p_i^α exist on the string, and transform as tensors as indicated by the i and α indices. $\overset{+}{\Pi}{}^{\alpha j}_{i\beta}\, p_j^\beta =: \overset{+}{p}{}_j^\alpha$ and $\overset{-}{\Pi}{}^{\alpha j}_{i\beta}\, p_j^\beta =: \overset{-}{p}{}_j^\alpha$ are defined to be J-antiholomorphic and J-holomorphic components of $p_j^\alpha = \overset{+}{p}{}_j^\alpha + \overset{-}{p}{}_j^\alpha$ respectively, where the complex-structure-compatibility projector is $\overset{+}{\Pi}{}^{i\beta}_{\alpha j} := \frac{1}{2}(\delta_\alpha^\beta \delta_j^i + \varepsilon_\alpha{}^\beta J^i{}_j)$, which annihilates the J-holomorphic component $\overset{-}{p}{}_j^\alpha$. Its complement is $\overset{-}{\Pi}{}^{i\beta}_{\alpha j} := \frac{1}{2}(\delta_\alpha^\beta \delta_j^i - \varepsilon_\alpha{}^\beta J^i{}_j)$, which annihilates the J-antiholomorphic component $\overset{+}{p}{}_j^\alpha$. These have the usual projector properties:

$$\overset{+}{\Pi}{}^{\alpha k}_{i\gamma}\,\overset{+}{\Pi}{}^{\gamma j}_{k\beta} = \overset{+}{\Pi}{}^{\alpha j}_{i\beta} \;,\quad \overset{-}{\Pi}{}^{\alpha k}_{i\gamma}\,\overset{-}{\Pi}{}^{\gamma j}_{k\beta} = \overset{-}{\Pi}{}^{\alpha j}_{i\beta} \quad \text{and} \quad \overset{+}{\Pi}{}^{\alpha k}_{i\gamma}\,\overset{-}{\Pi}{}^{\gamma j}_{k\beta} = 0 = \overset{-}{\Pi}{}^{\alpha k}_{i\gamma}\,\overset{+}{\Pi}{}^{\gamma j}_{k\beta} \tag{6.1}$$

$$\overset{+}{\Pi}{}^{\alpha j}_{i\beta}\,\overset{+}{p}{}_j^\beta = \overset{+}{p}{}_j^\alpha \;,\quad \overset{-}{\Pi}{}^{\alpha j}_{i\beta}\,\overset{+}{p}{}_j^\beta = 0 \;,\quad \overset{-}{\Pi}{}^{\alpha j}_{i\beta}\,\overset{-}{p}{}_j^\beta = \overset{-}{p}{}_j^\alpha \;,\quad \overset{+}{\Pi}{}^{\alpha j}_{i\beta}\,\overset{-}{p}{}_j^\beta = 0 \tag{6.2}$$

$\overset{-}{\Pi}{}^{\alpha j}_{i\beta}\,\overset{+}{p}{}_j^\beta = 0$ is the same as $\overset{+}{p}{}_j^\alpha J^j{}_i = -\overset{+}{p}{}_i^\beta \varepsilon_\beta{}^\alpha$, which expresses the J-antiholomorphicity condition of $\overset{+}{p}{}_j^\alpha$. $\overset{+}{p}{}_j^\alpha$ is the J-antiholomorphic component of p_j^α and $\overset{-}{p}{}_j^\alpha$ is the J-holomorphic component of p_j^α. $\overset{-}{p}{}_j^\alpha J^j{}_i = \overset{-}{p}{}_i^\beta \varepsilon_\beta{}^\alpha$, which is the same as $\overset{+}{\Pi}{}^{\alpha j}_{i\beta}\,\overset{-}{p}{}_j^\beta = 0$ and expresses the J-holomorphicity condition of $\overset{-}{p}{}_j^\alpha$. Note that $\varepsilon_\alpha{}^\beta = -\varepsilon^\beta{}_\alpha$ and $J^j{}_i = -J_i{}^j$ because of the metric compatibility condition of an almost hermitian manifold. The projectors are well defined on the configuration bundle and the multiphase space bundle (Indices are raised and lowered by the metric tensors).

Configuration space action

We define the configuration space action:

$$\begin{aligned}\mathcal{S}[u^i(\sigma^\alpha)] = \int_\Sigma \mathrm{d}^2\sigma\ \mathcal{L}(u^i, \partial_\alpha u^i, \sigma^\alpha) = \int \mathrm{d}^2\sigma\ \partial_\alpha u^i\ \overset{+}{\Pi}{}^{\alpha\beta}_{ij}\ \partial_\beta u^j \int_\Sigma \mathrm{d}^2\sigma \mid \overset{+}{\partial_\alpha} u^i \mid^2 = \\ = \int_\Sigma \mathrm{d}^2\sigma\ \frac{1}{2}(\ \partial_\alpha u^i g_{ij} h^{\alpha\beta} \partial_\beta u^j + \partial_\alpha u^i J_{ij} \varepsilon^{\alpha\beta} \partial_\beta u^j\ )\end{aligned} \tag{6.3}$$

Because the Lagrangian density is positive semi-definite, the minima in the action ($\mathcal{S} = 0$) occur when $\overset{+}{\partial_\alpha} u^i\ (\sigma) := \overset{+}{\Pi}{}^{\alpha j}_{i\beta}\ \partial_\alpha u^i\ (\sigma) = 0$ holds on the section $u^i(\sigma)$ of the configuration bundle. This is the definition of $u^i(\sigma^\alpha)$ being a J-holomorphic embedding, $u : \Sigma \longrightarrow M$, of the complex curve Σ in M. These minimum solutions remain minima under variations which preserve the J-holomorphicity. The Lagrangian density measures the deviation from J-holomorphicity: the anti-J-holomorphic projected part of the embedding.

The Euler-Lagrange equations for the Lagrangian (6.3) are:

$$0 \approx \mathrm{EL}_k = \partial_\alpha u^i\, (\partial_k\, \overset{+}{\Pi}{}^{\alpha\beta}_{ij}\,)\, \partial_\beta u^j - 2\, \partial_\alpha\, (\overset{+}{\Pi}{}^{\alpha\beta}_{kj}\, \partial_\beta u^j)$$

$$
\begin{aligned}
&= \partial_\alpha u^i \partial_\beta u^j \; h^{\alpha\beta} \; (\partial_k g_{ij} - \partial_j g_{ik} - \partial_i g_{kj}) \;+\; \partial_\alpha u^i \partial_\beta u^j \; \epsilon^{\alpha\beta} \; (\partial_k \omega_{ij} - \partial_j \omega_{ik} - \partial_i \omega_{kj}) \\
&\qquad -2 \; h^{\alpha\beta} g_{kj} \; \partial_\alpha \partial_\beta u^j \\
&= \partial_\alpha u^i \partial_\beta u^j \; h^{\alpha\beta} \; (-2) \Gamma^{k'}_{ij} g_{kk'} \;+\; \partial_\alpha u^i \partial_\beta u^j \; \epsilon^{\alpha\beta} \; \mathrm{d}\omega_{kij} \; - 2 \; h^{\alpha\beta} g_{kj} \; \partial_\alpha \partial_\beta u^j \\
&= \partial_\alpha u^i \partial_\beta u^j \; (-2 h^{\alpha\beta} \; \Gamma^{k'}_{ij} \; g_{kk'} \;+\; \epsilon^{\alpha\beta} \; \mathrm{d}\omega_{kij}) \; - 2 \; \partial^\alpha \partial_\alpha u^j g_{jk} \\
&= \partial_\alpha u^i \partial_\beta u^j \; \epsilon^{\alpha\beta} \; \mathrm{d}\omega_{kij} \; - 2 \, D^\alpha \partial_\alpha u^j g_{jk}
\end{aligned}
\tag{6.4}
$$

where

$$
D_\beta(\partial_\alpha u^j) \; := \; \partial_\beta(\partial_\alpha u^j) \;+\; \partial_\beta u^i \; \Gamma^j_{ij'} \; \partial_\alpha u^{j'} \quad \text{and} \quad \omega_{ij} := J_{ij} = g_{ij'} J^{j'}{}_i \tag{6.5}
$$

Thus the field equations are

$$
D^\alpha \partial_\alpha u^j g_{jk} \;\approx\; \frac{3}{2} \partial_\alpha u^i \partial_\beta u^j \; \epsilon^{\alpha\beta} \; \partial_{[k} \omega_{ij]} \tag{6.6}
$$

which is the wave equation with a source term. Note that $\partial_{[k}\omega_{ij]} = D_{[k}\omega_{ij]} = (\mathrm{d}\omega)_{kij}$ is the exterior derivative of the 2-form ω. If the target manifold is Almost-Kähler, $\mathrm{d}\omega = 0$, the source term is zero and the Euler-Lagrange equations simplify to $D^\alpha \partial_\alpha u^j \approx 0$. If u is J-holomorphic, $\overset{+}{\partial_\alpha} u^i \; (\sigma) \; := \; \overset{+}{\Pi}{}^{\alpha j}_{i\beta} \; \partial_\alpha u^i \; (\sigma) \; = \; 0$, then taking the covariant derivative of the latter and contracting we obtain $D^\alpha \partial_\alpha u^j g_{jk} \;\approx\; \frac{1}{2} \partial_\alpha u^i \partial_\beta u^j \; \epsilon^{\alpha\beta} \; (D_j \omega_{ik} + D_i \omega_{jk})$. The right hand side is equal to $\frac{3}{2} \partial_\alpha u^i \partial_\beta u^j \; \epsilon^{\alpha\beta} \; D_{[k}\omega_{ij]}$ because, if u is J-holomorphic, then $\partial_\alpha u^i \partial_\beta u^j \; \epsilon^{\alpha\beta} \; D_k \omega_{ij} = 0$. Thus we recover the equations of motion above.

Variations in the field configuration $u^i(\sigma^\alpha)$

For an infinitesimal variation of the field $\delta_\epsilon u^i \; = \; \epsilon^i(\sigma^\alpha)$, the change in the Lagrangian is

$$
\begin{aligned}
&\delta_\epsilon \mathcal{L} \;=\; \partial_\alpha u^i \, [2 \, \overset{+}{\Pi}{}^{\alpha\beta}_{ij} \;\; \partial_\beta \epsilon^j + \tfrac{1}{2} \epsilon^k (h^{\alpha\beta} \partial_k g_{ij} + \varepsilon^{\alpha\beta} \partial_k J_{ij}) \; \partial_\beta u^j \;] \;=\; \partial_\alpha u^i \, [2 \overset{+}{\partial^\alpha} \epsilon_i + \tfrac{1}{2} \epsilon^k (h^{\alpha\beta} 2 \Gamma_{k(ij)} + \\
&\varepsilon^{\alpha\beta} (D_k J_{ij} + \Gamma^{i'}_{ki} J_{i'j} + \Gamma^{j'}_{kj} J_{ij'}) \; \partial_\beta u^j \;] \;=\; \partial_\alpha u^i \, [2 \overset{+}{\partial^\alpha} \epsilon_i + \epsilon^k (h^{\alpha\beta} \Gamma_{kij} + \varepsilon^{\alpha\beta} (\tfrac{1}{2} D_k J_{ij} + \Gamma^{i'}_{ki} J_{i'j}) \; \partial_\beta u^j \;] \;= \\
&\partial_\alpha u^i \, [2 \overset{+}{\partial^\alpha} \epsilon_i + \epsilon^k (D_k (\tfrac{1}{2} \varepsilon^{\alpha\beta} J_{ij}) + \Gamma^{i'}_{ki} \; \overset{+}{\Pi}{}^{\alpha\beta}_{i'j} \;) \; \partial_\beta u^j \;] \;=\; \partial_\alpha u^i \, [2 \overset{+}{\partial^\alpha} \epsilon_i + \epsilon^k (D_k (\tfrac{1}{2} \varepsilon^{\alpha\beta} J_{ij}) + \Gamma^{i'}_{kj} \; \overset{-}{\Pi}{}^{\alpha\beta}_{i'i} \;) \; \partial_\beta u^j \;] \;= \\
&\partial_\alpha u^i \, [2 \overset{+}{\partial^\alpha} \epsilon_i + \epsilon^k (D_k (\tfrac{1}{2} \varepsilon^{\alpha\beta} J_{ij}) + \Gamma^{i'}_{kj} \; \overset{+}{\Pi}{}^{\alpha\beta}_{ii'} \;) \; \partial_\beta u^j \;] \;=\; \partial^\alpha u_i \, [2 \overset{+}{\partial_\alpha} \epsilon^i + \epsilon^k (D_k (\tfrac{1}{2} \varepsilon_\alpha{}^\beta J^i{}_j) + \Gamma^{i'}_{kj} \; \overset{+}{\Pi}{}^{i\beta}_{\alpha i'} \;) \; \partial_\beta u^j \;] \;= \\
&\partial^\alpha u_i \, [2 \overset{+}{\partial_\alpha} \epsilon^i + \epsilon^k (D_k \; \overset{+}{\Pi}{}^{i\beta}_{\alpha j} \; + \Gamma^{i'}_{kj} \; \overset{+}{\Pi}{}^{i\beta}_{\alpha i'} \;) \; \partial_\beta u^j \;] \;=\; \partial^\alpha u_i \, [2 \overset{+}{\partial_\alpha} \epsilon^i + \epsilon^k (\partial_k \; \overset{+}{\Pi}{}^{i\beta}_{\alpha j} \; + \Gamma^{i}_{ki'} \; \overset{+}{\Pi}{}^{i'\beta}_{\alpha j} \;) \; \partial_\beta u^j \;]
\end{aligned}
$$

$$
=: \; 2 \; \partial^\alpha u_i \;\; \overset{+}{D}{}^{\partial u, J}_\alpha \; \epsilon^i \tag{6.7}
$$

where the abbreviatied notation $\overset{+}{D}{}^{\partial u, J}_\alpha \; \epsilon^i$ is defined by the last equality.

We can write this as

$$
\begin{aligned}
\delta_\epsilon \mathcal{L} \; &= \delta_\epsilon u^k (\partial_\alpha u^i \partial_\beta u^j \epsilon^{\alpha\beta} \; \mathrm{d}\omega_{kij} - 2 D^\alpha \partial_\alpha u^j g_{jk}) \\
&-\partial_\alpha ((2 \partial_\beta u^i \overset{+}{\Pi}{}^{\alpha\kappa}_{ij} \; \partial_\kappa u^j - \delta^\alpha_\beta \partial_\lambda u^i \overset{+}{\Pi}{}^{\lambda\kappa}_{ij} \; \partial_\kappa u^j \;) \delta_\epsilon x^\beta \;\; - 2 \overset{+}{\Pi}{}^{\alpha\beta}_{ij} \; \partial_\beta u^j \delta_\epsilon u^i)
\end{aligned}
\tag{6.8}
$$

The Lagrangian is invariant, $\delta_\epsilon \mathcal{L} = 0$, under the variation, if the variation is such that the factor in square brackets above, $2\overset{+}{D}{}^{\partial u,J}_{\alpha}\ \epsilon^i$, is zero. We write

$$\overset{+}{\Pi}{}^{i\beta}_{\alpha j}\ \delta_\epsilon \partial_\beta u^j(\sigma)\ =:\ \overset{+}{\delta}_\epsilon(\partial_\alpha u^i)(\sigma)\ =\ \overset{+}{\partial}_\alpha \epsilon^i(\sigma) \tag{6.9}$$

Then the invariance condition $\overset{+}{D}{}^{\partial u,J}_{\alpha}\ \epsilon^i = 0$ is

$$\overset{+}{\partial}_\alpha \epsilon^i(\sigma) = \ -\frac{1}{2}\epsilon^k(D_k\ \overset{+}{\Pi}{}^{i\beta}_{\alpha j}\ + \Gamma^{i'}_{kj}\ \overset{+}{\Pi}{}^{i\beta}_{\alpha i'}\ )\ \partial_\beta u^j \tag{6.10}$$

which is the same as

$$\overset{+}{\partial}_\alpha \epsilon^i(\sigma) = \ -\ \frac{1}{2}\epsilon^k(\partial_k\ \overset{+}{\Pi}{}^{i\beta}_{\alpha j}\ +\ \Gamma^{i}_{ki'}\ \overset{+}{\Pi}{}^{i'\beta}_{\alpha j}\ )\ \partial_\beta u^j \tag{6.11}$$

We will now write the Witten variation of his projected multimomenta which is, when $\rho = 0$ (i.e. the odd fields are set zero), in his notation:

$$\begin{aligned}\delta_\epsilon \sigma^\alpha\ =\ 0\ \ ,\ \ \delta_\epsilon u^i\ =\ \epsilon^i\ \ ,\ \ \delta_\epsilon H^i_\alpha\ &=\ \epsilon^k\,(\frac{1}{2}\varepsilon_\alpha{}^\beta D_k J^i{}_j -\ \Gamma^i_{kj}\delta^\alpha_\beta)\,H^j_\beta\\ &=\ -\epsilon^k\,(\frac{1}{2}J^j{}_l D_k J^i{}_j +\ \Gamma^i_{kl}\ )\,H^l_\alpha\end{aligned} \tag{6.12}$$

where in his notation the positive-projection component $\overset{+}{p}{}^j_\beta$ is written as H^j_β. The negative-projection component $\overset{-}{p}{}^j_\beta$ is not present in the Witten model.

Rewriting the above in our notation:

$$\delta_\epsilon(\overset{+}{p}{}^i_\alpha)(\sigma)\ =\ \epsilon^k(D_k\ \overset{+}{\Pi}{}^{i\beta}_{\alpha j}\ -\ \Gamma^i_{ki'}\ \overset{+}{\Pi}{}^{i'\beta}_{\alpha j}\ )\ \overset{+}{p}{}^j_\beta \tag{6.13}$$

we can rearrange:

$$\Delta_\epsilon(\overset{+}{p}{}^i_\alpha)(\sigma)\ \ :=\ \delta_\epsilon(\overset{+}{p}{}^i_\alpha)(\sigma)\ +\ \epsilon^k\Gamma^i_{kj}\ (\overset{+}{p}{}^j_\beta)\ =\ \epsilon^k D_k\ \overset{+}{\Pi}{}^{i\beta}_{\alpha j}(\overset{+}{p}{}^j_\beta)\ =\ (\delta_\epsilon\ \overset{+}{\Pi}{}^{i\beta}_{\alpha j})\ \overset{+}{p}{}^j_\beta \tag{6.14}$$

so the covariant change, Δ_ϵ, in $\overset{+}{p}{}^j_\beta$ arising from a change $\delta_\epsilon u^i\ =\ \epsilon^i$ is simply the covariant change in the projector $\overset{+}{\Pi}{}^{i'\beta}_{\alpha j}$ acting on $\overset{+}{p}{}^j_\beta$. This was the variation which Witten chose to define his BRST-like variation. The other projected part of the multimomentum, $\overset{-}{p}{}^i_\alpha$, and the multimomentum itself, p^i_α, did not appear in Witten's model.

We can conjecture:

$$\Delta_\epsilon(\overset{-}{p}{}^i_\alpha)(\sigma)\ \ :=\ \delta_\epsilon(\overset{-}{p}{}^i_\alpha)(\sigma)\ +\ \epsilon^k\Gamma^i_{kj}\ (\overset{-}{p}{}^j_\beta)\ =\ \epsilon^k D_k\ \overset{-}{\Pi}{}^{i\beta}_{\alpha j}(\overset{-}{p}{}^j_\beta)\ =\ (\delta_\epsilon\ \overset{-}{\Pi}{}^{i\beta}_{\alpha j})\overset{-}{p}{}^j_\beta \tag{6.15}$$

Using,

$$\delta_\epsilon(\overset{+}{p}{}^i_\alpha)(\sigma)\ =\ \delta_\epsilon\overset{+}{\Pi}{}^{i\beta}_{\alpha j}\ p^j_\beta\ \ +\overset{+}{\Pi}{}^{i\beta}_{\alpha j}\ \delta_\epsilon\ p^j_\beta \tag{6.16}$$

we obtain the Witten variation re-expressed for comparison with (6.10):

$$\overset{+}{\Pi}{}^{i\beta}_{\alpha j}\ \delta_\epsilon(p^j_\beta)(\sigma)\ =:\overset{+}{\delta}_\epsilon(p^i_\alpha)(\sigma)\ =\ -\ \epsilon^k\,(\partial_k\ \overset{+}{\Pi}{}^{i\beta}_{\alpha j}\overset{-}{p}{}^j_\beta\ \ +\ \Gamma^i_{ki'}\ \overset{+}{\Pi}{}^{i'\beta}_{\alpha j}\ p^j_\beta\ ) \tag{6.17}$$

Variation of on-shell field configuration

If the on-shell condition, $\overset{+}{\partial}_\alpha u^i\,(\sigma) \;:=\; \overset{+}{\Pi}{}^{\alpha j}_{i\beta}\;\partial_\alpha u^i\,(\sigma) \;=\; 0$, holds, then

$$\begin{aligned}\delta_\epsilon \mathcal{L} \;&=\; \partial^\alpha u_i\,\epsilon^k\,(\partial_k\;\overset{+}{\Pi}{}^{i\beta}_{\alpha j})\;\partial_\beta u^j \;=\; \partial^\alpha u_i\,\epsilon^k\,(\partial_k\;(\overset{+}{\Pi}{}^{i\gamma}_{\alpha m}\overset{+}{\Pi}{}^{m\beta}_{\gamma j}\;))\;\partial_\beta u^j \\ &=\; \partial^\alpha u_i\,\epsilon^k\,(\;(\partial_k\overset{+}{\Pi}{}^{i\gamma}_{\alpha m}\overset{+}{\Pi}{}^{m\beta}_{\gamma j}\;)\;+\;(\overset{+}{\Pi}{}^{i\gamma}_{\alpha m}\partial_k\overset{+}{\Pi}{}^{m\beta}_{\gamma j}\;))\;\partial_\beta u^j \\ &=\; \partial^\alpha u_i\,\epsilon^k\;\;(\partial_k\overset{+}{\Pi}{}^{i\gamma}_{\alpha m}\overset{+}{\Pi}{}^{m\beta}_{\gamma j}\;)\;\;\partial_\beta u^j\;\;+\;\;\overset{+}{\partial}{}^\gamma u_m\;\;\epsilon^k\,(\overset{+}{\Pi}{}^{i\gamma}_{\alpha m}\partial_k\overset{+}{\Pi}{}^{m\beta}_{\gamma j}\;)\;\partial_\beta u^j \;=\; 0\end{aligned} \tag{6.18}$$

as expected for a variational extremum.

6.1.1 Multiphase-space Legendre transformation

As we are looking for a multiphase-space BRST formulation we now need to use multimomenta.

The multiphase-space Legendre transformation from the configuration bundle to the multiphase-space bundle with multimomenta:

$$p_i^\alpha \approx\; \frac{\partial \mathcal{L}}{\partial(\partial_\alpha u^i)} \;=\; 2\,\overset{+}{\Pi}{}^{\alpha\beta}_{ij}\;\;\partial_\beta u^j = 2\;\frac{1}{2}(h^{\alpha\beta}g_{ij} + \varepsilon^{\alpha\beta}J_{ij})\partial_\beta u^j =: 2\overset{+}{\partial^\alpha} u_i \tag{6.19}$$

the projector property $\overset{-}{\Pi}{}^{\gamma i}_{k\alpha}\overset{+}{\Pi}{}^{\alpha\beta}_{ij} = 0$ leads to primary constraints on the negative projected parts of the multimomenta:

$$\overset{-}{p_i^\alpha}\;\;:= \overset{-}{\Pi}{}^{\alpha j}_{i\beta}\;p_j^\beta =\; \frac{1}{2}(\delta_\beta^\alpha\delta_i^j - \varepsilon^\alpha{}_\beta J_i{}^j)p_j^\beta \approx\; 0. \tag{6.20}$$

DDW Hamiltonian

The DeDonder-Weyl Hamiltonian is

$$\begin{aligned}\mathcal{H} = \;p_i^\alpha\partial_\alpha u^i - \mathcal{L} \;&=\; \frac{1}{4}\;\overset{+}{p_i^\alpha}\;\overset{+}{p_\alpha^i}\;\;+\overset{-}{p_i^\alpha}\partial_\alpha u^i \;\;=\; \frac{1}{4}\;\overset{+}{p_i^\alpha}\;h_{\alpha\beta}\;g^{ij}\;\overset{+}{p_j^\beta}\;\;-\;\;\partial_\alpha\overset{-}{p_i^\alpha}\;u^i \\ &=\; \frac{1}{4}\;p_i^\alpha\;\overset{+}{\Pi}{}^{ij}_{\alpha\beta}\;p_j^\beta\;\;+\partial_\beta(p_i^\alpha\;\overset{-}{\Pi}{}^{i\beta}_{\alpha j})\;u^j \\ &=\; \frac{1}{8}\;p_i^\alpha(h_{\alpha\beta}g^{ij} + \varepsilon_{\alpha\beta}J^{ij})\;p_j^\beta\;\;+\partial_\beta(\;p_i^\alpha\frac{1}{2}(\delta_\alpha^\beta\delta_j^i - \varepsilon_\alpha{}^\beta J^i{}_j)\;)\;u^j\end{aligned} \tag{6.21}$$

where we repeat the definition of the projected parts of the multimomenta:

$$\overset{+}{p_i^\alpha}\;\;:= \overset{+}{\Pi}{}^{\alpha j}_{i\beta}\;p_j^\beta =\; \frac{1}{2}(\delta_\beta^\alpha\delta_i^j + \varepsilon^\alpha{}_\beta J_i{}^j)p_j^\beta\;\;\text{and}\;\;\overset{-}{p_i^\alpha}\;\;:= \overset{-}{\Pi}{}^{\alpha j}_{i\beta}\;p_j^\beta =\; \frac{1}{2}(\delta_\beta^\alpha\delta_i^j - \varepsilon^\alpha{}_\beta J_i{}^j)p_j^\beta \tag{6.22}$$

and have employed integration by parts in the first line, assuming that the DDW Hamiltonian is a term in a multiphase space Lagrangian, as below.

Multiphase-space action

A first order formalism multiphase-space action would be

$$\mathcal{S}_{MP} = \int \mathrm{d}^2\sigma \; p_i^\alpha \partial_\alpha u^i \; - \; \mathcal{H} = \int \mathrm{d}^2\sigma \; p_i^\alpha \partial_\alpha u^i \; - \; (\; \frac{1}{4} \; \overset{+}{p}{}_i^\alpha \; \overset{+}{p}{}^i_\alpha \; + \overset{-}{p}{}_i^\alpha \partial_\alpha u^i \;)$$

$$= \int \mathrm{d}^2\sigma \; \overset{+}{p}{}_i^\alpha \partial_\alpha u^i \; - \; \frac{1}{4} \; \overset{+}{p}{}_i^\alpha \; \overset{+}{p}{}^i_\alpha \; = \int \mathrm{d}^2\sigma \;\; \partial_\alpha u^i \; \overset{+}{\Pi}{}_{i\beta}^{\alpha j} \; p_j^\beta \; - \; \frac{1}{4} \;\; p^i_\alpha \; \overset{+}{\Pi}{}_{i\beta}^{\alpha j} \; p_j^\beta \tag{6.23}$$

Another action with the same Euler-Lagrange equations is:

$$\mathcal{S}_{MP2} = \int \mathrm{d}^2\sigma \;\; \partial_\alpha u^i \; \overset{+}{\Pi}{}_{i\beta}^{\alpha j} \; p_j^\beta \; - \; \frac{1}{4} \;\; p^i_\alpha p_i^\alpha \tag{6.24}$$

The following action has Euler-Lagrange equation for p: $\partial_\alpha u^i \; - \; \frac{1}{4}\overset{-}{p}{}^i_\alpha \;\; \approx \;\; 0$ which implies $\overset{+}{\partial_\alpha u^i} \; (\sigma) \approx \; 0$, which is the equation of motion of the original Lagrangian (6.3).

$$\mathcal{S}_{MP3} = \int \mathrm{d}^2\sigma \;\; \partial_\alpha u^i p_i^\alpha \; - \; \frac{1}{4} \;\; p^i_\alpha \; \overset{-}{\Pi}{}_{i\beta}^{\alpha j} \; p_j^\beta \tag{6.25}$$

When the Euler-Lagrange equation for p is substituted into (6.25) the right hand side is identically zero.

Variation of the multiphase-space action

From (3.51) this is

$$\delta\mathcal{S}_{MP} = \int \mathrm{d}^2\sigma \; \delta\overset{+}{p}{}_i^\alpha \; (\; \partial_\alpha u^i \; - \; \frac{1}{2} \; \overset{+}{p}{}_i^\alpha \;) \; + \; \overset{+}{p}{}_i^\alpha \; (\; \delta\partial_\alpha u^i \; - \; \frac{1}{4} \; \delta u^k \, \partial_k \, \overset{+}{\Pi}{}_{\alpha\beta}^{ij} \;\; p_j^\beta \;) =$$

$$\int \mathrm{d}^2\sigma \; \delta\overset{+}{p}{}_i^\alpha \; (\; \partial_\alpha u^i \; - \; \frac{1}{2} \; \overset{+}{p}{}_i^\alpha \;) \; - \; \delta u^k \; (\; \partial_\alpha p_k^\alpha \; + \; \frac{1}{4} \; p_i^\alpha \; \partial_k \; \overset{+}{\Pi}{}_{\alpha\beta}^{ij} \;\; p_j^\beta \;) \; + \; \delta\sigma^\gamma \; (\; \frac{1}{4} \; p_i^\alpha \; \partial_\gamma \; \overset{+}{\Pi}{}_{\alpha\beta}^{ij} \;\; p_j^\beta \;)$$

$$+ \; \partial_\alpha [\; \delta u^i \overset{+}{p}{}_i^\alpha \; + \; (\; \partial_\alpha u^i \; \overset{+}{\Pi}{}_{i\beta}^{\alpha j} \; p_j^\beta \; - \; \frac{1}{4} \; p_i^\alpha \; \overset{+}{\Pi}{}_{\alpha\beta}^{ij} \, p_j^\beta \;) \;] \tag{6.26}$$

6.1.2 Gauge variations

Witten gives the following infinitesimal variation of the fields (setting the odd parity fields to zero) , with variational parameter $\epsilon^i(\sigma)$ (Note however that, unlike for us, his $\overset{+}{p}{}_i^\alpha$ are not projected parts of multimomenta but independent variables):

$$\delta_\epsilon\sigma^\alpha \; = \; 0 \;\; , \;\; \delta_\epsilon u^i(\sigma) \; = \; \epsilon^i(\sigma) \;\; , \;\; \delta_\epsilon \; \overset{+}{p}{}_i^\alpha(\sigma) \; = \; \epsilon^k(\sigma) \;\; (\frac{1}{2}\varepsilon^\alpha{}_\beta(\sigma) \; D_k J_i{}^l(u) \; + \; \delta^\alpha{}_\beta(\sigma) \; \Gamma_{ki}{}^l(u) \;) \, \overset{+}{p}{}_l^\beta$$

$$=: \;\; \epsilon^k \, \Gamma_{k|i\beta}^{\;\;\alpha l}(\sigma, u) \;\; \overset{+}{p}{}_l^\beta$$

$$= \; -\epsilon^k \, (\; \frac{1}{2} \; J_j{}^l \; D_k J_i{}^j \; - \; \Gamma_{ki}{}^l \;\;) \; \overset{+}{p}{}_l^\alpha \;\; = \; -\epsilon^k \, (\; \frac{1}{2} \; D_k J_i{}^j \; + \; \Gamma_{ki}{}^r \; J_r{}^j \;\;) \; J_j{}^l \, \overset{+}{p}{}_l^\alpha \tag{6.27}$$

because $\varepsilon^{\alpha}{}_{\beta}\overset{\pm}{p}{}^{\beta}_{i} = \mp J_i{}^j \overset{\pm}{p}{}^{\alpha}_{j}$. and (conjecture)

$$\delta_\epsilon \overset{-}{p}{}^{\alpha}_{i} = \epsilon^k \left(\frac{1}{2}\varepsilon^{\alpha}{}_{\beta} D_k J_i{}^l + \delta^{\alpha}{}_{\beta}\Gamma_{ki}{}^l\right)\overset{-}{p}{}^{\beta}_{l} =: \epsilon^k\, \Gamma^{\alpha l}_{k|i\beta}\overset{-}{p}{}^{\beta}_{l} \tag{6.28}$$

then (conjecture)

$$\delta_\epsilon p^{\alpha}_{i} = \epsilon^k \left(\frac{1}{2}\varepsilon^{\alpha}{}_{\beta} D_k J_i{}^l + \delta^{\alpha}{}_{\beta}\Gamma_{ki}{}^l\right) p^{\beta}_{l} =: \epsilon^k\, \Gamma^{\alpha l}_{k|i\beta} p^{\beta}_{l} \tag{6.29}$$

and then (conjecture)

$$\delta_\epsilon p^{\alpha}_{i} = \epsilon^k \left(\frac{1}{2}\delta^{\alpha}{}_{\beta} J_{l'}{}^l D_k J_i{}^{l'} + \delta^{\alpha}{}_{\beta}\Gamma_{ki}{}^l\right) p^{\beta}_{l} \tag{6.30}$$

$$\delta_\epsilon \overset{+}{p}{}^{\alpha}_{i} = -\{\epsilon^j \overset{-}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_\beta = \epsilon^j \frac{1}{2}\varepsilon^{\alpha}{}_{\beta} D_j J_i{}^l\, \overset{+}{p}{}^{\beta}_{l} = \epsilon^j\left(-\frac{1}{2}D_j J_i{}^m J_m{}^l\right)\overset{+}{p}{}^{\alpha}_{l} = \epsilon^j \bar{C}_{ij}{}^l\, \overset{+}{p}{}^{\alpha}_{l} \tag{6.31}$$

$$\delta_\epsilon \overset{-}{p}{}^{\alpha}_{i} = -\{\epsilon^j \overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_\beta = \epsilon^j \frac{1}{2}\varepsilon^{\alpha}{}_{\beta} D_{[i} J_{j]}{}^l\, \overset{-}{p}{}^{\beta}_{l} = \epsilon^j \frac{1}{2}D_{[i} J_{j]}{}^m J_m{}^l\, \overset{-}{p}{}^{\alpha}_{l} = \epsilon^j C_{ij}{}^l\, \overset{-}{p}{}^{\alpha}_{l} \tag{6.32}$$

where $C_{ij}{}^l(u) := \frac{1}{2}D_{[i}J_{j]}{}^m J_m{}^l = \frac{1}{8}(\overset{-}{N}_{ij}{}^l - \overset{+}{N}_{ij}{}^l)$ and $\bar{C}_{ij}{}^l(u) := -\frac{1}{2}D_j J_i{}^m J_m{}^l$ are the structure functions. See appendix A.

First, defining the J-covariant derivative along the string world surface as

$$D^J_\alpha := \partial_\alpha + \Gamma_{kj}{}^i \partial_\alpha u^j + \frac{1}{2}\,\varepsilon_\alpha{}^\beta\; D_k J^i{}_j \partial_\beta u^j =: \partial_\alpha + \Gamma^{\beta i}_{k|j\alpha}\partial_\beta u^j =: \partial_\alpha + \overset{u}{\Gamma}{}^{i}_{k|\alpha}, \tag{6.33}$$

then the variation using (6.27) of the first order multiphase-space Lagrangian (6.1.1) is

$$\begin{aligned}
\delta_\epsilon \mathcal{L} &= \delta_\epsilon\left(\overset{+}{p}{}^{\alpha}_{i}\partial_\alpha u^i - \frac{1}{4}\overset{+}{p}{}^{\alpha}_{i}\overset{+}{p}{}^{i}_{\alpha}\right) = \delta_\epsilon \overset{+}{p}{}^{\alpha}_{i}\left(\partial_\alpha u^i - \frac{1}{2}\overset{+}{p}{}^{i}_{\alpha}\right) + \overset{+}{p}{}^{\alpha}_{i}\left(\partial_\alpha \epsilon^i + \frac{1}{2}\epsilon^k\,\Gamma_{kj}{}^i \overset{+}{p}{}^{j}_{\alpha}\right) = \\
&\overset{+}{p}{}^{\alpha}_{i}\partial_\alpha\epsilon^i + \epsilon^k\left(\frac{1}{2}\varepsilon^{\alpha}{}_{\beta}D_k J_i{}^j\left(\partial_\alpha u^i - \frac{1}{2}\overset{+}{p}{}^{i}_{\alpha}\right) + \partial_\alpha u^i \Gamma_{ki}{}^j\right)\overset{+}{p}{}^{\beta}_{j} = \\
&\overset{+}{p}{}^{\alpha}_{k} D^J_\alpha \epsilon^k - \epsilon^k \frac{1}{4}\varepsilon^{\alpha}{}_{\beta} D_k J_i{}^j\, \overset{+}{p}{}^{i}_{\alpha}\overset{+}{p}{}^{\beta}_{j} = \overset{+}{p}{}^{\alpha}_{k} D^J_\alpha \epsilon^k - \epsilon^k \frac{1}{2} D_k \overset{+}{\Pi}{}^{\alpha j}_{i\beta}\, \overset{+}{p}{}^{i}_{\alpha}\overset{+}{p}{}^{\beta}_{j} = \\
&\overset{+}{p}{}^{\beta}_{j}\left(D^J_\beta \epsilon^j - \epsilon^k \frac{1}{2}D_k\overset{+}{\Pi}{}^{\alpha j}_{i\beta}\overset{+}{p}{}^{i}_{\alpha}\right) = p^{\beta'}_{j'}\overset{+}{\Pi}{}^{\beta j'}_{j\beta'}\left(D^J_\beta\epsilon^j - \epsilon^k\frac{1}{2}D_k\overset{+}{\Pi}{}^{\alpha j}_{i\beta}\overset{+}{p}{}^{i}_{\alpha}\right) = p^{\beta'}_{j'}\overset{+}{\Pi}{}^{\beta j'}_{j\beta'} D^J_\beta\epsilon^j = \\
&\overset{+}{p}{}^{\beta}_{j} D^J_\beta \epsilon^j := p^{\beta'}_{j'}\overset{+}{\Pi}{}^{\alpha j'}_{i\beta'}\left(\partial_\alpha\epsilon^i + \epsilon^k\partial_\alpha u^j\,\Gamma_{kj}{}^i - \epsilon^k\frac{1}{2}\partial_\alpha u^l\; D_k J_j{}^i J_l{}^j\right)
\end{aligned} \tag{6.34}$$

so $\delta_\epsilon\mathcal{L} = \overset{+}{p}{}^{\beta}_{j} D^J_\beta \epsilon^j$. This implies that $\delta_\epsilon \mathcal{L} = 0$ if $\overset{+}{D}{}^{Jj}_{\beta i}\epsilon^i := \overset{+}{\Pi}{}^{\alpha j}_{i\beta} D^J_\alpha \epsilon^i = 0$ - the positive projected part of the J-covariant derivative of the variation is zero. So there is a gauge freedom of variations where the variation only has negative projected J-covariant derivative.

6.1.3 Multimomentum algebra

In this section the multi-Poisson brackets on multiphase space between the projected multimomenta for the topological sigma model are calculated. Use is made of the Nijenhuis and conjugate Nihenhuis tensors described in appendix A.

Brackets between projected multimomenta:

$$\{\overset{+}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} = \{\overset{+}{\Pi}{}^{\beta k}_{j\gamma}\, p^{\gamma}_{k}\,,\ \overset{+}{\Pi}{}^{\alpha k}_{i\gamma}\, p^{\gamma}_{k}\}_{\beta} = \overset{+}{\Pi}{}^{\beta k}_{j\gamma}\, p^{\gamma}_{k}\ \cdot (\frac{\overleftarrow{\partial}}{\partial u^{i}} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\beta}_{i}}) \cdot \overset{+}{\Pi}{}^{\alpha k}_{i\gamma}\, p^{\gamma}_{k}$$

$$= -\frac{1}{2}\ [\ \partial_{[i} J_{j]}{}^{m} J_{m}{}^{l}\ (\varepsilon^{\alpha}{}_{\beta} J_{l}{}^{l'})\ +\ \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ (\delta^{\alpha}{}_{\beta} \delta_{l}{}^{l'})\ ]\ p^{\beta}_{l'}$$

$$= -\frac{1}{2}\ [\ \partial_{[i} J_{j]}{}^{m} J_{m}{}^{l}\ (\overset{+}{p}{}^{\alpha}_{l}\ -\ \overset{-}{p}{}^{\alpha}_{l})\ +\ \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ (\overset{+}{p}{}^{\alpha}_{l}\ +\ \overset{-}{p}{}^{\alpha}_{l})\ ]$$

$$= -\ \frac{1}{4}\ (\ \overset{+}{N}_{ij}{}^{l}\, \overset{-}{p}{}^{\alpha}_{l}\ -\ (\overset{-}{N}_{ij}{}^{l}\ +\ 4 J_{[i}{}^{r}\, \Gamma_{j]r}{}^{m} J_{m}{}^{l}\,)\overset{+}{p}{}^{\alpha}_{l}\,) \tag{6.35}$$

$$\{\overset{-}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} = \frac{1}{2}\ [\partial_{(i} J_{j)}{}^{m} J_{m}{}^{l}\ (\varepsilon^{\alpha}{}_{\beta} J_{l}{}^{l'}) + \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ (\delta^{\alpha}{}_{\beta} \delta_{l}{}^{l'})\ ]\ \ p^{\beta}_{l'}$$

$$= \frac{1}{2}\ [\ \partial_{(i} J_{j)}{}^{m} J_{m}{}^{l}\ (\overset{+}{p}{}^{\alpha}_{l}\ -\ \overset{-}{p}{}^{\alpha}_{l})\ +\ \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ (\overset{+}{p}{}^{\alpha}_{l}\ +\ \overset{-}{p}{}^{\alpha}_{l})\ ]$$

$$= \frac{1}{2}\ [(\ \partial_{(i} J_{j)}{}^{m} J_{m}{}^{l} +\ \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ )\, \overset{+}{p}{}^{\alpha}_{l}\ + (\ -\partial_{(i} J_{j)}{}^{m} J_{m}{}^{l} +\ \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ )\, \overset{-}{p}{}^{\alpha}_{l}] \tag{6.36}$$

$$\{\overset{+}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} = -\frac{1}{2}\ [\partial_{(i} J_{j)}{}^{m} J_{m}{}^{l}\ (\varepsilon^{\alpha}{}_{\beta} J_{l}{}^{l'}) - \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ (\delta^{\alpha}{}_{\beta} \delta_{l}{}^{l'})\ ]\ p^{\beta}_{l'}$$

$$= -\frac{1}{2}\ [\ \partial_{(i} J_{j)}{}^{m} J_{m}{}^{l}\ (\overset{+}{p}{}^{\alpha}_{l}\ -\ \overset{-}{p}{}^{\alpha}_{l})\ -\ \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ (\overset{+}{p}{}^{\alpha}_{l}\ +\ \overset{-}{p}{}^{\alpha}_{l})\ ]$$

$$= -\frac{1}{2}\ [(\ \partial_{(i} J_{j)}{}^{m} J_{m}{}^{l} -\ \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ )\, \overset{+}{p}{}^{\alpha}_{l}\ + (\ -\partial_{(i} J_{j)}{}^{m} J_{m}{}^{l} -\ \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ )\, \overset{-}{p}{}^{\alpha}_{l}] \tag{6.37}$$

The following bracket is the constraint algebra:

$$\{\overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} = \frac{1}{2}\ [\ \partial_{[i} J_{j]}{}^{m} J_{m}{}^{l}\ (\ \varepsilon^{\alpha}{}_{\beta} J_{l}{}^{l'})\ -\ \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ (\delta^{\alpha}{}_{\beta} \delta_{l}{}^{l'})\ ]\ p^{\beta}_{l'}$$

$$= \frac{1}{2}\ [\ \partial_{[i} J_{j]}{}^{m} J_{m}{}^{l}\ (\overset{+}{p}{}^{\alpha}_{l}\ -\ \overset{-}{p}{}^{\alpha}_{l})\ -\ \partial_{m} J_{[j}{}^{l} J_{i]}{}^{m}\ (\overset{+}{p}{}^{\alpha}_{l}\ +\ \overset{-}{p}{}^{\alpha}_{l})\ ]$$

$$= -\ \frac{1}{4}\ (\ \overset{+}{N}_{ij}{}^{l}\, \overset{+}{p}{}^{\alpha}_{l}\ +\ (\overset{-}{N}_{ij}{}^{l}\ +\ 4 J_{[i}{}^{r}\, \Gamma_{j]r}{}^{m} J_{m}{}^{l}\,)\, \overset{-}{p}{}^{\alpha}_{l}\,) \tag{6.38}$$

where $\overset{+}{N}_{ij}{}^{l}, \overset{-}{N}_{ij}{}^{l}$ are the Nijenhuis and conjugate Nijenhuis tensors respectively:

$$\overset{+}{N}_{ik}{}^{j} = 2(D_{[i} J^{j}{}_{\bar{k}]} + D_{[\bar{i}} J^{j}{}_{k]}) = (D_{i} J^{j}{}_{k'} J^{k'}{}_{k} - D_{k} J^{j}{}_{k'} J^{k'}{}_{i} + J^{i'}{}_{i} D_{i'} J^{j}{}_{k} - J^{i'}{}_{k} D_{i'} J^{j}{}_{i}) \tag{6.39}$$

where D is any derivative. In particular we can have

$$\overset{+}{N}_{ij}{}^{l}\ =\ 2(\partial_{m} J_{[j}{}^{l} J_{i]}{}^{m} - \partial_{[i} J_{j]}{}^{m} J_{m}{}^{l}\ ) \tag{6.40}$$

The Nijenhuis tensor is the Frolicher-Nijenhuis bracket of the complex structure tensor (viewed as a vector valued one form) with itself:

$$\overset{+}{N} = \frac{1}{2}\{J, J\}_{FN} \tag{6.41}$$

The conjugate Nijenhuis tensor is:

$$\overset{-}{N}_{ij}{}^{l} \;=\; 2(\partial_m J^{\,l}_{[j} J_{i]}{}^{m} + \partial_{[i} J_{j]}{}^{m} J^{\,l}_{m}\,) - 4 J^{\,r}_{[i}\,\Gamma_{j]r}{}^{m} J_m{}^{l} \tag{6.42}$$

Adding the above, we obtain, as expected,

$$\{p^{\beta}_{j}, p^{\alpha}_{i}\}_{\beta} \;=\; \{\overset{+}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} \;+\; \{\overset{-}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} \;+\; \{\overset{+}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} + \{\overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; 0 \tag{6.43}$$

and

$$\{\overset{-}{p}{}^{\beta}_{j}, p^{\alpha}_{i}\}_{\beta} = \{\overset{-}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} \;+\; \{\overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} = \;\partial_i J_j{}^{m} J^{\,l}_{m}\;(\,\overset{+}{p}{}^{\alpha}_{l} - \overset{-}{p}{}^{\alpha}_{l}\;\;) \tag{6.44}$$

$$\{\overset{+}{p}{}^{\beta}_{j}, p^{\alpha}_{i}\}_{\beta} = \{\overset{+}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} \;+\; \{\overset{+}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} = \;-\;\partial_i J_j{}^{m} J^{\,l}_{m}\;(\,\overset{+}{p}{}^{\alpha}_{l} - \overset{-}{p}{}^{\alpha}_{l}\;\;) \tag{6.45}$$

We would like to have $\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m} \;= \partial_m J^{\,l}_{[j} J_{i]}{}^{m}$ in order to factorize these factors on the right hand side in the above (6.35) and (6.38) so as to obtain the closure of the constraint algebra:

$$\{\overset{+}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; -\frac{1}{2}\,[\,\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m}\,(\varepsilon^{\alpha}{}_{\beta} J_l{}^{l'} + \delta^{\alpha}{}_{\beta}\delta_l{}^{l'}\,)\,]\,p^{\beta}_{l'} \;=\; -\,\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m}\,\overset{+}{\Pi}{}^{\alpha l'}_{l\beta}\;p^{\beta}_{l'} \;=\; -\,\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m}\,\overset{+}{p}{}^{\alpha}_{l} \tag{6.46}$$

$$\{\overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; -\frac{1}{2}\,[\,\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m}\,(-\varepsilon^{\alpha}{}_{\beta} J_l{}^{l'} + \delta^{\alpha}{}_{\beta}\delta_l{}^{l'}\,)\,]\,p^{\beta}_{l'} \;=\; -\,\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m}\,\overset{-}{\Pi}{}^{\alpha l'}_{l\beta}\;p^{\beta}_{l'} \;=\; -\,\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m}\,\overset{-}{p}{}^{\alpha}_{l} \tag{6.47}$$

The condition above, which leads to the closing of the constraint algebra, $\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m} = \partial_m J^{\,l}_{[j} J_{i]}{}^{m}$, is the condition that the Nijenhuis tensor is zero, and this is the condition that the almost hermitian manifold M is hermitian and J is a complex structure. See appendix A for the different types of almost hermitian manifolds. We now specialize to three classes of almost-Hermitian manifold with extra structure:

1) For M a *Hermitian* manifold, $\overset{+}{N}_{ij}{}^{l} = 0$, $\overset{-}{N}_{ij}{}^{l} = 4(\,\partial_m J^{\,l}_{[j} J_{i]}{}^{m} - J^{\,r}_{[i}\,\Gamma_{j]r}{}^{m} J_m{}^{l}\,) = 4(\,\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m} - J^{\,r}_{[i}\,\Gamma_{j]r}{}^{m} J_m{}^{l}\,)$, and,

$$\{\overset{+}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; -\,\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m}\,\overset{+}{p}{}^{\alpha}_{l} \;=\; \frac{1}{4}\,(\,\overset{-}{N}_{ij}{}^{l} + 4 J^{\,r}_{[i}\,\Gamma_{j]r}{}^{m} J_m{}^{l}\,)\,\overset{+}{p}{}^{\alpha}_{l} \tag{6.48}$$

and

$$\{\overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; -\,\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m}\,\overset{-}{p}{}^{\alpha}_{l} \;=\; -\frac{1}{4}\,(\,\overset{-}{N}_{ij}{}^{l} + 4 J^{\,r}_{[i}\,\Gamma_{j]r}{}^{m} J_m{}^{l}\,)\,\overset{-}{p}{}^{\alpha}_{l} \tag{6.49}$$

$$\{\overset{+}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; \frac{1}{2}\,(\,\partial_i J_j{}^{m} J^{\,l}_{m}\,\overset{-}{p}{}^{\alpha}_{l} - \partial_j J_i{}^{m} J^{\,l}_{m}\,\overset{+}{p}{}^{\alpha}_{l}\,) \text{ and } \{\overset{-}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; \frac{1}{2}\,(\partial_i J_j{}^{m} J^{\,l}_{m}\,\overset{+}{p}{}^{\alpha}_{l} - \partial_j J_i{}^{m} J^{\,l}_{m}\,\overset{-}{p}{}^{\alpha}_{l}\,) \tag{6.50}$$

$$\{\overset{+}{p}{}^{\beta}_{j}, p^{\alpha}_{i}\}_{\beta} \;=\; -\partial_i J_j{}^{m} J^{\,l}_{m}\,p^{\alpha}_{l} \;\text{ and }\; \{\overset{-}{p}{}^{\beta}_{j}, p^{\alpha}_{i}\}_{\beta} \;=\; \partial_i J_j{}^{m} J^{\,l}_{m}\,p^{\alpha}_{l} \tag{6.51}$$

2) For M a *Quasi-Kähler* manifold, $\overset{-}{N}_{ij}{}^{l} = 0$, $\overset{+}{N}_{ij}{}^{l} = 4\,\partial_m J^{\,l}_{[j} J_{i]}{}^{m} - 4 J^{\,r}_{[i}\,\Gamma_{j]r}{}^{m} J_m{}^{l} = -4\,\partial_{[i} J_{j]}{}^{m} J^{\,l}_{m} + 4 J^{\,r}_{[i}\,\Gamma_{j]r}{}^{m} J_m{}^{l}$, and,

$$\{\overset{+}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; \partial_{[i} J_{j]}{}^{m} J^{\,l}_{m}\,\overset{-}{p}{}^{\alpha}_{l} \;=\; -\frac{1}{4}\,\overset{+}{N}_{ij}{}^{l}\,\overset{-}{p}{}^{\alpha}_{l} + J^{\,r}_{[i}\,\Gamma_{j]r}{}^{m} J_m{}^{l}\,\overset{+}{p}{}^{\alpha}_{l} \tag{6.52}$$

and

$$\{\overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; \partial_{[i}J_{j]}{}^{m} J_{m}{}^{l}\, \overset{+}{p}{}^{\alpha}_{l} \;=\; -\,\frac{1}{4}\; \overset{+}{N}_{ij}{}^{l}\; \overset{+}{p}{}^{\alpha}_{l} - J_{[i}{}^{r}\,\Gamma_{j]r}{}^{m} J_{m}{}^{l}\, \overset{-}{p}{}^{\alpha}_{l} \tag{6.53}$$

$$\{\overset{+}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; \frac{1}{2}\,(\, -\partial_{i}J_{j}{}^{m}J_{m}^{\,l}\; \overset{+}{p}{}^{\alpha}_{l} \;+\; \partial_{j}J_{i}{}^{m}J_{m}^{\,l}\; \overset{-}{p}{}^{\alpha}_{l}\;) \quad \text{and} \tag{6.54}$$

$$\{\overset{-}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; \frac{1}{2}\,(\, -\, \partial_{i}J_{j}{}^{m}J_{m}^{\,l}\; \overset{-}{p}{}^{\alpha}_{l} \;+\; \partial_{j}J_{i}{}^{m}J_{m}^{\,l}\; \overset{+}{p}{}^{\alpha}_{l}\;) \tag{6.55}$$

$$\{\overset{+}{p}{}^{\beta}_{j}, p^{\alpha}_{i}\}_{\beta} \;=\; -\,\frac{1}{2}\,\partial_{i}J_{j}{}^{m}J_{m}^{\,l}\,(\,\overset{+}{p}{}^{\alpha}_{l} \,-\, \overset{-}{p}{}^{\alpha}_{l}\,) \;\; \text{and} \;\; \{\overset{-}{p}{}^{\beta}_{j}, p^{\alpha}_{i}\}_{\beta} \;=\; \frac{1}{2}\,\partial_{i}J_{j}{}^{m}J_{m}^{\,l}\,(\,\overset{+}{p}{}^{\alpha}_{l} \,-\, \overset{-}{p}{}^{\alpha}_{l}\,) \tag{6.56}$$

so:

$$-\delta_{\epsilon}\overset{-}{p}{}^{\alpha}_{i} \;=\; \epsilon^{j}\partial_{[i}J_{j]}{}^{m}J_{m}^{\,l}\; \overset{+}{p}{}^{\alpha}_{l} \quad , \quad -\,\delta_{\epsilon}\overset{+}{p}{}^{\alpha}_{i} \;=\; \epsilon^{j}\frac{1}{2}\,(\, -\, \partial_{i}J_{j}{}^{m}J_{m}^{\,l}\; \overset{-}{p}{}^{\alpha}_{l} \;+\; \partial_{j}J_{i}{}^{m}J_{m}^{\,l}\; \overset{+}{p}{}^{\alpha}_{l}\;) \tag{6.57}$$

$$\text{and} \qquad -\,\delta_{\epsilon}p^{\alpha}_{i} \;=\; \epsilon^{j}\frac{1}{2}\;\partial_{i}J_{j}{}^{m}J_{m}^{\,l}\;(\;\overset{+}{p}{}^{\alpha}_{l} \;-\; \overset{-}{p}{}^{\alpha}_{l}\;) \tag{6.58}$$

where the variations are generated by $\overset{-}{p}{}^{\alpha}_{i}$.

3) For M a Kähler manifold, $\overset{+}{N}_{ij}{}^{l} = 0 = \overset{-}{N}_{ij}{}^{l}$ and

$$\{\overset{+}{p}{}^{\beta}_{j}, \overset{+}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; J_{[i}{}^{r}\,\Gamma_{j]r}{}^{m}J_{m}{}^{l}\,\overset{+}{p}{}^{\alpha}_{l} \tag{6.59}$$

and

$$\{\overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_{\beta} \;=\; -J_{[i}{}^{r}\,\Gamma_{j]r}{}^{m}J_{m}{}^{l}\,\overset{-}{p}{}^{\alpha}_{l} \tag{6.60}$$

which is the desired first class constraint algebra. But the Kähler condition is too strong.

The Hermitian condition gives a first class algebra on the constraints but again the condition is too restrictive because in this case the J-holomorphic curves are holomorphic. The Quasi-Kähler condition is desirable because there are useful relations between the J tensors and the covariant derivative D as indicated in appendix A. But as shown above, under the Quasi-Kähler condition, the primary constraint algebra does not close on the primary constraints.

6.2 Comparison with the Witten variations

We now compare the above brackets for the Quasi-Kahler condition with the BRST-like variations defined in the Witten model as explained in section 6.4.

In the Witten paper, the positive projected part of the multimomentum $\overset{+}{p}{}^{\alpha}_{i} \;:=\; \overset{+}{\Pi}{}^{\alpha j}_{i\beta}\; p^{\beta}_{j}$ is an independent variable and has the role of the multimomentum and is written as H^{α}_{i}. The 'body' terms (setting the grassmann odd fields to zero) of his super-multiphase-space BRST-like action is:

$$\mathcal{S}_{WP0} = \int \mathrm{d}^{2}\sigma\; H^{\alpha}_{i}\partial_{\alpha}u^{i} \;-\; \frac{1}{4}\; H^{\alpha}_{i}\; H^{i}_{\alpha} \tag{6.61}$$

And the BRST-like variation he defines is, setting the BRST-like ghost variable $\rho = 0$, and replacing the BRST variation parameter $i\epsilon\chi^i$ by the gauge variation parameter ϵ^i, so as to obtain the conjectured original gauge variation:

$$\delta_\epsilon u^i \;=\; \epsilon^i \quad , \quad \delta_\epsilon\, H_i^\alpha \;=\; \epsilon^k\,(\frac{1}{2}\varepsilon^\alpha{}_\beta D_k J_i{}^j + \Gamma^j_{ki}\delta^\alpha_\beta)H^\beta_j \tag{6.62}$$

We will now compare these variations defined by Witten with the variations generated by the primary constraint $\overset{-}{p}{}^\gamma_j \approx 0$ in our model, via the multi-Poisson bracket.

The multi-Poisson bracket in the multiphase space is, as usual:

$$\{f,g\}_\alpha := -df \llcorner \Pi_\alpha \lrcorner dg = f \cdot (\frac{\overleftarrow{\partial}}{\partial u^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha}) \cdot g = \left(\frac{\partial f}{\partial u^i}\right)\left(\frac{\partial g}{\partial p_i^\alpha}\right) - \left(\frac{\partial f}{\partial p_i^\alpha}\right)\left(\frac{\partial g}{\partial u^i}\right) \tag{6.63}$$

In terms of multi-Poisson brackets, the variation of an observable O, generated by the constraint $\overset{-}{p}{}^\gamma_i$, is:

$$\delta_\epsilon\; O \;:=\; -\,\{\;\epsilon^i\;\overset{-}{p}{}^\gamma_i\;,\;O\;\}_\gamma \tag{6.64}$$

For $O = u^k$ this is

$$\delta_\epsilon\; u^k \;\;=\;\; -\;\{\;\epsilon^j\;\overset{-}{p}{}^\gamma_j\;,\;u^k\;\}_\gamma = \epsilon^j\;\overset{-}{\Pi}{}^{\gamma k}_{j\gamma} = \frac{1}{2}\;\epsilon^j\;(\;\delta^\gamma_\gamma\delta^k_j\;\;-\;\varepsilon^\gamma{}_\gamma J_j{}^k\;) \tag{6.65}$$

$$= \frac{1}{2}\;\epsilon^j\;(\,2\,\delta^k_j\;\;-\;\;0\,J_j{}^k\,) = \;\epsilon^k \tag{6.66}$$

This is the same as the Witten variation above.

This also shows that the $\overset{-}{p}{}^\gamma_j$ generator variation parameter ϵ^j is the same as the variation of u^i, and this implies that this generator parameter must satisfy the $\delta_\epsilon u^k$ variation equation (6.10) in our model above.

We now consider $\delta_\epsilon H_i^\alpha$:

We replace Witten's notation for the positive projected multimomenta, H_i^α, with our notation, $\overset{+}{p}{}^\alpha_i$, to make the comparison easier. The conjectured gauge variation above derived from Witten's variation is then:

$$-\delta_\epsilon \overset{+}{p}{}^\alpha_i \;\;=\;\; \epsilon^j\;(\;\frac{1}{2}\;D_j J_i{}^m\;J^{\,l}_m\;\;-\;\;\Gamma_{ji}{}^l\;\;)\;\overset{+}{p}{}^\alpha_l \tag{6.67}$$

In our model we have, for a Quasi-Kahler case:

$$-\delta_\epsilon \overset{+}{p}{}^\alpha_i \;\;=\;\; \{\epsilon^j \overset{-}{p}{}^\beta_j, \overset{+}{p}{}^\alpha_i\}_\beta = \;\frac{1}{2}\;\epsilon^j\;(\;-\;\partial_i J_j{}^m J^{\,l}_m\;\overset{-}{p}{}^\alpha_l\;\;+\;\;\partial_j J_i{}^m J^{\,l}_m\;\overset{+}{p}{}^\alpha_l\;) \tag{6.68}$$

which we calculated in (6.55) above.

The difference between the Witten variation and (6.55) is that the Witten variation employs the covariant derivative D_j instead of ∂_j, has a Christoffel symbol, Γ, term and assumes $\overset{-}{p}{}^{\beta}_{j} = 0$, although (6.53) gives

$$-\delta_\epsilon \overset{-}{p}{}^{\alpha}_{i} \;=\; \{\epsilon^j \overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_\beta \;=\; \epsilon^j \; \{\overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_\beta \;\;=\; \epsilon^j \; \partial_{[i} J_{j]}{}^m J_m^{\;l} \; \overset{+}{p}{}^{\alpha}_{l} \;=\; -\;\frac{1}{4}\;\epsilon^j \; \overset{+}{N}_{ij}{}^{l} \; \overset{+}{p}{}^{\alpha}_{l} \tag{6.69}$$

(assuming the primary constraints $\overset{-}{p}{}^{\beta}_{j} \approx 0$), which would then not preserve the primary constraints $\overset{-}{p}{}^{\beta}_{j} \approx 0$. This would lead to a contradiction with the Witten model where the negative projected multimomentum $\overset{-}{p}{}^{\beta}_{j}$ does not appear.

The structure functions (if were to have $\overset{-}{p}{}^{\beta}_{j}$ instead of $\overset{+}{p}{}^{\beta}_{j}$) are $C_{ij}{}^{l} \;=\; -\;\frac{1}{4}\;\overset{+}{N}_{ij}{}^{l}$, where $\overset{+}{N}_{ij}{}^{l}$ is the Nijenhuis tensor. Note that the derivative of the almost complex structure tensor in the Nijenhuis tensor can be any derivation, not only the covariant derivative.

We conjecture that the D_j instead of ∂_j issue could be resolved by treating ∂_j, which arises from the multi-Poisson bracket action, as being a covariant derivative when acting on tensors like p_j. For this the ∂_j in the definition of the multi-Poisson bracket should be replaced by the covariant derivative D_j. This may also remove the Γ term in (6.53) so as to give the desired constraint algebra $\{\overset{-}{p}{}^{\beta}_{j}, \overset{-}{p}{}^{\alpha}_{i}\}_\beta \;=\; -\;\frac{1}{4}\;\overset{+}{N}_{ij}{}^{l}\;\overset{-}{p}{}^{\alpha}_{l}$

6.3 Multiphase-space BRST formulation of the model

In the same manner that we approached the multiphase-space BRST formulation of the example models in section (4.7) we start by replacing the gauge variation parameter ϵ^i by a reverse grassmann parity (in this case grassmann odd) ghost field η^i. The multiphase space is enlarged to a super-multiphase space B, where ρ^α_i are the grassmann odd multimomenta conjugate to η^i. This super-multiphase space has the structure of Riemannian multiphase space of chapter 5.

The objective is to add a gauge 'fixing' term $\;\overset{-}{p}{}^{\alpha}_{i}\;\partial_\alpha u^i - \frac{1}{2}\;\overset{-}{p}{}^{\alpha}_{i}\;\overset{-}{p}{}^{i}_{\alpha}\;$ + ghost terms to the first order Lagrangian (6.1.1) in such a way that it remains invariant under the nilpotent global BRST variation δ_B on the super-multiphase space $B = \{u^i, p^\alpha_i; \eta^i, \rho^\alpha_i\}$, where δ_B is the multi-Hamiltonian vector field generated by, perhaps, the BRST ($d-1$ -form) charge, $J^\alpha\,\mathrm{d}^{d-1}x_\alpha \;=\; -\mathrm{d}^{d-1}x_\alpha\,(\eta^i \overset{-}{p}{}^{\alpha}_{i} - \frac{1}{2}\overset{-}{\rho}{}^{\alpha}_{i} C^i_{jk}\eta^j\eta^k)$, via the supermulti-brackets: $\delta_\epsilon\Phi = -i\epsilon\frac{1}{d}\{J^\alpha, \Phi\}_\alpha$.

The multi-Poisson bracket in the Riemannian super-multiphase space is (5.2) in section 5.2:

$$\{f, g\}_\alpha := f\overleftarrow{\mathrm{d}} \llcorner \Pi_\alpha \lrcorner \overrightarrow{\mathrm{d}} g = -\mathrm{d}f \llcorner \Pi_\alpha \lrcorner \mathrm{d}g = f \cdot (\frac{\overleftarrow{\partial}}{\partial u^i} \wedge \frac{\overrightarrow{\partial}}{\partial p^\alpha_i}) \cdot g \;-\; f \cdot (\frac{\overleftarrow{\partial}}{\partial \eta^i} \wedge \frac{\overrightarrow{\partial}}{\partial \rho^\alpha_i}) \cdot g \;+$$

$$\Gamma^j_{ik}\,\eta^k f\cdot(\frac{\overleftarrow{\partial}}{\partial\eta^j}\wedge\frac{\overrightarrow{\partial}}{\partial p^\alpha_i})\cdot g\;-\;\Gamma^j_{ik}\,\rho^\gamma_j f\cdot(\frac{\overleftarrow{\partial}}{\partial\rho^\alpha_k}\wedge\frac{\overrightarrow{\partial}}{\partial p^\gamma_i})\cdot g\;-\;\frac{1}{2}R_{ijk}{}^l\,\rho^\gamma_l\,\eta^k f\cdot(\frac{\overleftarrow{\partial}}{\partial p^\gamma_j}\wedge\frac{\overrightarrow{\partial}}{\partial p^\alpha_i})\cdot g$$

$$= f\cdot(\frac{\overleftarrow{\partial}}{\partial u^i}\wedge\frac{\overrightarrow{\partial}}{\partial p^\alpha_i})\cdot g\;-\;f\cdot(\frac{\overleftarrow{D}}{D\eta^i}\wedge\frac{\overrightarrow{D}}{D\rho^\alpha_i})\cdot g\;-\Gamma^l_{ik,j}\,\rho^\gamma_l\,\eta^k f\cdot(\frac{\overleftarrow{\partial}}{\partial p^\gamma_j}\wedge\frac{\overrightarrow{\partial}}{\partial p^\alpha_i})\cdot g \tag{6.70}$$

where

$$\frac{\overrightarrow{D}}{D\rho^\alpha_i}:=\frac{\overrightarrow{\partial}}{\partial\rho^\alpha_i}+\Gamma^i_{jk}\,\eta^k\frac{\overrightarrow{\partial}}{\partial p^\alpha_j}\quad\text{and}\quad\frac{\overleftarrow{D}}{D\eta^i}:=\frac{\overleftarrow{\partial}}{\partial\eta^i}-\frac{\overleftarrow{\partial}}{\partial p^\alpha_j}\,\Gamma^k_{ij}\,\rho^\alpha_k \tag{6.71}$$

Following the multiphase-space BRST procedure, we conjecture that the SUSY variation could be generated from the conserved current (6.83) $J^\alpha=\bar{p}^\alpha_i\eta^i+\frac{1}{2}\varepsilon^\alpha{}_\gamma\rho^\gamma_i D_k J^i{}_j\eta^k\eta^j$ or $J^\alpha=\bar{p}^\alpha_i\eta^i-\frac{1}{2}\rho^\alpha_{i'}J^{i'}{}_i D_k J^i{}_j\eta^k\eta^j$ and the multiphase-space bracket above:

$$\delta_\epsilon\Phi=-i\epsilon\frac{1}{d}\{J^\alpha,\Phi\}_\alpha=-i\epsilon\frac{1}{d}\{\eta^i\bar{p}^\alpha_i-\frac{1}{2}\rho^\alpha_i C^i_{jk}\eta^j\eta^k,\Phi\}_\alpha \tag{6.72}$$

The bracket with the current J^α gives:

$$\delta_\epsilon u^i=-i\frac{1}{2}\epsilon\{J^\alpha,u^i\}_\alpha=i\epsilon\eta^i \tag{6.73}$$

$$\delta_\epsilon\eta^i=-i\frac{1}{2}\epsilon\{J^\alpha,\eta^i\}_\alpha=-i\frac{1}{2}\epsilon\,(\frac{1}{2}\varepsilon^\alpha{}_\alpha D_k J^i{}_j\eta^k\eta^j+\Gamma^i_{ik}\eta^k\eta^i\delta^\alpha_\alpha)=0 \tag{6.74}$$

this is zero because both terms in the bracket are zero. The first because $\varepsilon^\alpha{}_\alpha=0$, and the second because the product of odd variables $\eta^k\eta^i$ is antisymmetric in ik whereas the Christoffel symbol is symmetric in ik. This reproduces the Witten variations for u^i and η.

6.4 The Witten topological sigma model

This section presents and discusses the Witten model Lagrangian and its BRST-like appearance.

We have assumed that the Witten model is based on a σ model where the base space is a Riemann surface Σ with hermitian metric h and compatible complex structure ε and the target space is a d-dimensional almost hermitian manifold M with almost complex structure J and compatible metric g. A field configuration is a section $u:\Sigma\mapsto M$ and is a Riemann surface embedded in M and can be regarded as a string propagating in the manifold M. The target manifold M is such that it has an almost complex structure, which imposes limitations on possible topologies, but any almost complex manifold can be given a compatible metric. Because the (quantized) model is about finding global topological invariants, the restriction on metrics is not an obstacle for this. The Lagrangian measures the deviation from J-holomorphicity of the embedding.

Local coordinates on Σ are denoted by $(\sigma^\alpha\ ;\ \alpha = 0,1)$ and on M by $(u^i\ ;\ i = 1 \ldots d)$.

Our conjecture is that the Witten model is a type of multiphase space BRST construction. We now present the Witten model as it is in his paper.

In his paper [30], Witten constructs his model by first defining a set of fields $\Phi = (u^i, \chi^i, \rho_i^\alpha, H_i^\alpha)$, where $i = 1 \ldots \dim M$, $\alpha = 0, 1$, on a Riemann surface base space Σ. A field configuration $u^i(\sigma^\alpha)$ corresponds to an embedding of a two dimensional surface Σ parametrized by local coordinates $\sigma^\alpha, \alpha = 1, 2$, into an almost hermitian manifold M with local coordinates $u^i, i = 1, \ldots, \dim M$, together with some extra fields $(\chi^i, \rho_i^\alpha, H_i^\alpha)$ on the Riemann surface, which transform as tensors as suggested by the indices. H_i^α is bosonic with ghost number 0 and (χ^i, ρ_i^α) are fermionic and have ghost number 1 and -1 respectively. H_i^α and ρ_i^α are defined to be projected parts $H_i^\alpha = \overset{+}{\Pi}{}_{i\beta}^{\alpha j}\ H_j^\beta$ and $\rho_i^\alpha = \overset{+}{\Pi}{}_{i\beta}^{\alpha j}\ \rho_j^\beta$ where the almost complex-structure-compatibility projector is $\overset{+}{\Pi}{}_{\alpha j}^{i\beta} = \frac{1}{2}(\delta_\alpha^\beta \delta_j^i + \varepsilon_\alpha{}^\beta J^i{}_j)$. This means that $H_j^\alpha J^j{}_i = -H_i^\beta \varepsilon_\beta{}^\alpha$, the J-antiholomorphicity condition. He then writes down a global scalar fermionic infinitesimal variation $\delta\Phi$ with ghost number 1 for each of the fields Φ in such a way that this infinitesimal variation is constructed to have the nilpotent property $\delta\delta\Phi = 0$ for all of the fields. With this he then writes down an action, $\mathcal{S}_{MP} = \int \mathcal{L}\ \mathrm{d}^2\sigma = \delta \int \mathrm{d}^2\sigma\ Z$, where $Z = \rho_i^\alpha(\partial_\alpha u^i - \frac{1}{4} H_\alpha^i)$ has the appearance of a gauge fixing fermion. This action is conformally invariant because Z is conformally invariant, and δ- invariant because $\delta\mathcal{L} = \delta\delta Z = 0$, from the nilpotence property of δ. This action is (eqn. (2.14) in [30]):

$$\mathcal{S}_{WP} = \int \mathrm{d}^2\sigma\ H_i^\alpha \partial_\alpha u^i - \frac{1}{4} H_i^\alpha H_j^\beta h_{\alpha\beta} g^{ij} + (-i\rho_i^\alpha) D_\alpha^J \chi^i - \frac{1}{8} \chi^k \chi^l K_{kl}{}^{ij} \rho_i^\alpha \rho_j^\beta h_{\alpha\beta} \tag{6.75}$$

where

$$(-i\rho_i^\alpha) D_\alpha^J \chi^i \ :=$$

$$(-i\rho_i^\alpha)\ (\partial_\alpha \chi^i + \partial_\alpha u^k\ \Gamma_{kj}^i\ \chi^j + \frac{i}{2}\ \varepsilon_\alpha{}^\beta\ \partial_\beta u^j\ D_k J^i{}_j\ \chi^k) =\ (-i\rho_i^\alpha)\ (\partial_\alpha \chi^i\ +\ \partial_\beta u^j\ \Gamma_{k|j\alpha}^{\ \beta i}\ \chi^k\ ) \tag{6.76}$$

and where

$$\Pi_{i\beta|k}^{\alpha j}\ :=\ \Gamma_{k|i\beta}^{\ \alpha j}\ :=\ \frac{1}{2}\ \varepsilon^\alpha{}_\beta D_k J_i{}^j\ +\ \delta^\alpha{}_\beta \Gamma_{ki}{}^j \quad , \tag{6.77}$$

$$D_\alpha^J := \partial_\alpha + \partial_\alpha u^k\ \Gamma_{kj}{}^i + \frac{i}{2}\ \varepsilon_\alpha{}^\beta\ \partial_\beta u^j\ D_k J^i{}_j = \partial_\alpha +\ \partial_\beta u^j\, \Gamma_{k|j\alpha}^{\ \beta i} =: \partial_\alpha +\ \overset{u}{\Gamma}{}^{\ \ i}_{k|\alpha}, \tag{6.78}$$

is the ‘J-covariant derivative’.

and where

$$K_{kl}{}^{ij}\ :=\ R_{kl}{}^{ij} + \frac{1}{2} g^{mn} (D_k J_m{}^i)(D_l J_n{}^j) \tag{6.79}$$

is the ‘almost complex curvature’ tensor. So that D_α^J is a covariant derivative on the string world-surface which preserves the projection. The field H_i^α is an auxiliary field and, via the Euler-Lagrange equation for H_i^α , can be eliminated resulting in the following action (eqn. (2.16) in [30]):

$$\mathcal{S}_W =$$

$$\int \mathrm{d}^2\sigma\ \partial_\alpha u^i \overset{+\alpha\beta}{\Pi_{ij}}\ \partial_\beta u^j + (-i\rho_i^\alpha) D_\alpha^J \chi^i - \frac{1}{8}\chi^k\chi^l({R_{kl}}^{ij} + \frac{1}{2}g^{mn}(D_k {J_m}^i)(D_l {J_n}^j))\rho_i^\alpha\rho_j^\beta h_{\alpha\beta} \quad (6.80)$$

where the kinetic term $\partial_\alpha u^i \overset{+\alpha\beta}{\Pi_{ij}}\ \partial_\beta u^j$ only depends on the positive projected part of the velocities $\overset{+i\beta}{\Pi_{\alpha j}}\ \partial_\beta u^j$ via the projector $\overset{+i\beta}{\Pi_{\alpha j}} = \frac{1}{2}(\delta_\alpha^\beta\delta_j^i + {\varepsilon_\alpha}^\beta {J^i}_j)$.

Hrabak [84] pointed out that H_i^α and ρ_i^α had the role of multimomenta as can be seen from the Lagrangian in (6.75) which has the form of a super-multiphase-space Lagrangian, and Witten pointed out the BRST-like form of the fermionic symmetry δ: the action (6.80) has the form of a multiphase-space action with a DDW Hamiltonian given by the terms quadratic in the H's and the χ's and ρ's:

$$\mathcal{H}_W = \frac{1}{4}H_i^\alpha H_j^\beta h_{\alpha\beta} g^{ij} + \frac{1}{8}\chi^k\chi^l({R_{kl}}^{ij} + \frac{1}{2}g^{mn}(D_k {J_m}^i)(D_l {J_n}^j))\rho_i^\alpha\rho_j^\beta h_{\alpha\beta} \quad (6.81)$$

The fermionic part of the Lagrangian in (6.75) has the appearance of ghost terms required in a functional integral to produce the measure via gaussian integrals of quadratic ghost terms. The bosonic terms could be viewed as a gauge fixing terms, fixing to a J-holomorphic curve.

It should be noted that Witten's definition of H_i^α and ρ_i^α are projected parts $H_i^\alpha = \overset{+\alpha j}{\Pi_{i\beta}}\ p_j^\beta$ and $\rho_i^\alpha = \overset{+\alpha j}{\Pi_{i\beta}}\ \pi_j^\beta$ of the multimomenta p_j^β and π_j^β paired with u^j and χ^j respectively. The projection is preserved by a term containing DJ and the covariance of the i indicies is ensured by a term containing Γ_{kj}^i in the variation δ. These two terms results in the terms quadratic in the χ^i's in the Lagrangians (6.75) and (6.80) above when δ acts on Z. We have already described the Lagrangian above as having a gauge fixing term.

The Witten model action $\mathcal{S}_{WP}$ (6.75) above has the appearance of a BRST construction starting from the highly symmetric action:

$$\mathcal{S} = \int \mathrm{d}^2\sigma\ \partial_\alpha u^i \overset{+\alpha\beta}{\Pi_{ij}}\ \partial_\beta u^j \quad (6.82)$$

symmetric under variations $\delta u^i = \epsilon^i(\sigma^\alpha)$, which preserve the J-antiholomorphic part of the embedding $u^i(\sigma^\alpha)$ as described in section 6.1. The additional terms in the Lagrangian density (6.80) have the appearance of a Faddeev-Popov gauge fixing term and the Gaussian integral expansion of a functional determinant factor in a functional integral. Witten constructs the Lagrangian from δ, a nilpotent, odd, ghost number 1 vector field on a space $\mathbb{M}$ like a super-multiphase space (with only positive projected multimomenta however), with local coordinates $(u^i, \chi^i, \rho_i^\alpha, H_i^\alpha, \sigma^\alpha)_{i=1\ldots\dim \mathrm{M}\ ,\ \alpha=1,2}$, which has the properties of a BRST variation. Z above has the appearance of a gauge fixing fermion with H_i^α in the role of Lagrange multipliers enforcing the constraint on $\partial_\alpha u^i$, and ρ_i^α the ghosts (multi-) momenta. In the BRST construction the $\chi^i(\sigma)$ would be the parity-reversed gauge variation parameters.

Witten also gives the conserved current on Σ induced by the SUSY variation δ:

$$J^\alpha = H_i^\alpha\chi^i + \frac{1}{2}\rho_s^\alpha {J_i}^s D_k {J^i}_j \chi^k\chi^j \quad (6.83)$$

which has the appearance of a BRST-like charge, where H_i^α would have the role of the constraints and $J_i{}^s D_{[k} J^i{}_{j]}$ the role of the structure 'constants' of the Poisson brackets for first class constraints.

Because the target manifold is an almost Hermitian manifold, it is necessary to use certain symmetries of the gradient of the almost complex structure tensor in the calculations. In fact the model requires an extra condition on the covariant derivative of the almost complex structure which will make the almost complex manifold into an Almost- or Quasi-Kähler manifold. For this reason appendix A summarizes the properties of almost Hermitian manifolds, manifolds with an almost complex structure and a compatible metric. In particular the relevant properties of almost Hermitian manifolds with extra structure, such as Kähler, Almost Kähler, and Quasi-Kähler are briefly reviewed.

The BRST-like variation specified by Witten is, with fermionic parameter ϵ ,

$$\delta_\epsilon u^i = i\epsilon\chi^i \quad , \quad \delta_\epsilon \sigma^\alpha = 0 \quad , \quad \delta_\epsilon \chi^i = 0 \tag{6.84}$$

$$\delta_\epsilon \rho_i^\alpha = \epsilon\,(\ H_i^\alpha + \frac{i}{2}\varepsilon^\alpha{}_\beta D_k J_i{}^j \chi^k \rho_j^\beta + i\Gamma^j_{ki}\chi^k\rho_j^\alpha\ ) \tag{6.85}$$

$$\delta_\epsilon H_i^\alpha \ = $$
$$\epsilon(\frac{i}{2}\ \varepsilon^\alpha{}_\beta D_k J_i{}^j \chi^k H_j^\beta + i\Gamma^j_{ki}\chi^k H_j^\alpha - \frac{1}{4}\chi^k\chi^l(R_{kli}{}^j + R_{kl}{}^{i'j'} J_{ii'} J^j{}_{j'} + g^{mn}(D_k J_{mi})(D_l J_n{}^j))\rho_j^\alpha) \tag{6.86}$$

We rewrite these variations using slightly different notation, replacing Witten's $-i\chi^j$ with η^j, which was our notation, and keeping in mind that the Witten multimomentum ρ is the positive projected part of our ρ in previous sections:

$$\delta_\epsilon u^i = -\epsilon\eta^i \tag{6.87}$$

$$\delta_\epsilon \eta^i = 0 \tag{6.88}$$

$$\delta_\epsilon \rho_i^\alpha = \epsilon\,(H_i^\alpha - \frac{1}{2}\varepsilon^\alpha{}_\beta D_k J_i{}^j \eta^k \rho_j^\beta - \Gamma^j_{ki}\eta^k\rho_j^\alpha)\ = \epsilon\,(H_i^\alpha - \ \eta^k\,\Gamma_{k|i\beta}^{\ \alpha j}\,\rho_j^\beta) \tag{6.89}$$

$$\delta_\epsilon H_i^\alpha =$$
$$\epsilon\,(\ -\frac{1}{2}\ \varepsilon^\alpha{}_\beta D_k J_i{}^j \eta^k H_j^\beta - \Gamma^j_{ki}\eta^k H_j^\alpha + \frac{1}{4}\eta^k\eta^l(R_{kli}{}^j + R_{kl}{}^{i'j'} J_{ii'} J^j{}_{j'} + g^{mn}(D_k J_{mi})(D_l J_n{}^j))\rho_j^\alpha\ )$$
$$= -\epsilon\,(\ \eta^k\,\Gamma_{k|i\beta}^{\ \alpha j}\,H_j^\beta\ -\ \frac{1}{4}\eta^k\eta^l\bar{K}_{kli}{}^j\rho_j^\alpha\ ) \tag{6.90}$$

where

$$\Pi^{\alpha j}_{i\beta|k} := \Gamma_{k|i\beta}^{\ \alpha j} :=\ \frac{1}{2}\ \varepsilon^\alpha{}_\beta D_k J_i{}^j\ +\ \delta^\alpha{}_\beta \Gamma_{ki}{}^j \tag{6.91}$$

6.4.1 Concluding remarks and conjecture

In section 6.1, we have analysed the dynamics of the sigma model of J-holomorphic embeddings. We have calculated the gauge symmetries of our model and found that the negative projected multimomentum is the constraint, and have calculated the multi-Poisson brackets of the projected multimomenta for different subclasses of almost Hermitian structures on the target space, in the attempt to obtain a constraint algebra which closes. In the previous chapter 5, we set up a plausible candidate BRST super-multiphase space constructed from a Riemannian manifold target space, where we set up super-multi-Poisson brackets. We additionally considered an additional almost Hermitian structure J, and calculated the super-multi-Poisson brackets of J-holomorphic and J-antiholomorphic projected parts of the multimomenta. In this chapter we then applied the super-multi-Poisson brackets obtained in the previous chapter to the sigma model of J-holomorphic embedding having constructed a BRST current J^α. The BRST current has a term which incorporates the structure constant of the constriant algebra. The weak point in our analysis is not having established the correct structure constant previously, so we have conjectured a plausible structure constant.

Our project has been to attempt to obtain the Witten model as a multiphase-space BRST constructed from the sigma model of J-holomorphic embeddings. In more detail, the conjecture is that the multi-Poisson bracket in the Riemannian super-multiphase space, (5.2) in section 5.2 or some variation of it, together with the current (6.83) $J^\alpha = \overset{-}{p}{}_i^\alpha \eta^i + \frac{1}{2}\varepsilon^\alpha{}_\gamma \rho_i^\gamma D_k J^i{}_j \eta^k \eta^j$ or $J^\alpha = \overset{-}{p}{}_i^\alpha \eta^i - \frac{1}{2}\rho_{i'}^\alpha J^{i'}{}_i D_k J^i{}_j \eta^k \eta^j$ generates the multiphase-space BRST variation which is the same as the BRST-like variation Witten defines in his model at the end of the last section. The conjecture is also that the bracket of the BRST generator with itself is zero:

$$\{J^\alpha, J^{\alpha'}\}_\alpha = \{\overset{-}{p}{}_i^\alpha \eta^i + \frac{1}{2}\varepsilon^\alpha{}_\gamma \rho_i^\gamma D_k J^i{}_j \eta^k \eta^j \ , \ \overset{-}{p}{}_{i'}^{\alpha'} \eta^{i'} + \frac{1}{2}\varepsilon^{\alpha'}{}_\gamma \rho_{i'}^\gamma D_{k'} J^i{}_j \eta^k \eta^j\}_\gamma = 0 \tag{6.92}$$

This would be the required $\{Q, Q\} = 0$ property of the BRST observable.

Another conjecture is that the term containing (6.79)

$$K_{kl}{}^{ij} \ := \ R_{kl}{}^{ij} + \frac{1}{2} g^{mn} (D_k J_m{}^i)(D_l J_n{}^j) \tag{6.93}$$

in the Witten BRST-like Lagrangian, which arises from the $\rho_i^\alpha \delta_\epsilon(-\frac{1}{4}H_\alpha^i)$ part of the variation of the gauge fixing fermion can be produced by

$$\overset{+}{\Pi}{}_{i\gamma}^{\alpha k} \rho_k^\gamma \{ \ J^\beta \ , \ -\frac{1}{4} \overset{+}{\Pi}{}_{\alpha k}^{i\gamma} p_\gamma^k \ \}_\beta \tag{6.94}$$

The result (5.28) in the previous chapter suggests something like this may be the case.

Appendix A

Identities of the almost complex structure J

This appendix presents from [37] certain relationships between covariant derivatives of the almost complex structure tensors on almost Hermitian manifolds, which define subclasses of almost Hermitian manifolds, some of which are relevant to the Witten model. There are sixteen classes of almost Hermitian manifolds, that is, manifolds with an almost complex structure and a compatible metric. The classes here are defined by the symmetries of the covariant derivative of the Kähler form F: $D_i F_{jk} := D_i(J_j^{k'} g_{k'k})$. Gray [37] describe these classes in detail. In [5] he gives identities in these classes for the curvature tensor. Note that in 4 dimensions the number of classes reduces to four: Almost Hermitian, Hermitian, Almost Kähler, Kähler.

Given any (1,1)- tensor J (i.e. matrix or vector valued 1-form), the Nijenhuis tensor $\overset{+}{N}$ is a (2,1)-tensor: $\langle \overset{+}{N}(X,Y), \omega \rangle = \overset{+}{N}_{ik}{}^{j} X^i Y^k \omega_j$ where

$$\overset{+}{N}(X,Y) = -[X,Y] - J[JX,Y] + J[JY,X] + [JX,JY] \tag{A.1}$$

or equivalently $\overset{+}{N} = \frac{1}{2}[J,J]_{FN}$, where $[J,J]_{FN}$ is the Nijenhuis-Frolicher bracket of J with itself.

In tensor index notation, where barred indices are defined below (A.5):

$$\overset{+}{N}_{ik}{}^{j} \;=\; 2(D_{[i}J^{j}{}_{\bar{k}]} + D_{[\bar{i}}J^{j}{}_{k]}) \;=\; D_i J^j{}_{k'} J^{k'}{}_k - D_k J^j{}_{k'} J^{k'}{}_i + J^{i'}{}_i D_{i'} J^j{}_k - J^{i'}{}_k D_{i'} J^j{}_i \tag{A.2}$$

The derivative D above can be any derivative. Any Christoffel symbols cancel out.

Conjugate Nijenhuis tensor $\langle \bar{N}(X,Y),\omega\rangle \;=\; \bar{N}_{ik}{}^{j}X^iY^k\omega_j$ where

$$\bar{N}_{ik}{}^{j} := 2(-D_{[i}J^{j}{}_{\bar{k}]} + D_{[\bar{i}}J^{j}{}_{k]}) \;=\; -D_iJ^{j}{}_{k'}J^{k'}{}_{k} + D_kJ^{j}{}_{k'}J^{k'}{}_{i} + J^{i'}{}_{i}D_{i'}J^{j}{}_{k} - J^{i'}{}_{k}D_{i'}J^{j}{}_{i} \quad \text{(A.3)}$$

Christoffel symbols do not cancel here unlike the case for the Nijenhuis tensor above:

$$\bar{N}_{ij}{}^{l} \;:=\; 2(\partial_m J_{[j}^{\;l}J_{i]}^{\;m} + \partial_{[i}J_{j]}^{\;m}J_{m}^{\;l}\;) - 4J_{[i}^{\;r}\,\Gamma_{j]r}{}^{m}J_{m}{}^{l} = 2(\partial_{[\bar{i}}J^{l}{}_{j]} - \partial_{[i}J^{l}{}_{\bar{j}]}) - 4\,\Gamma_{[j\bar{i}]}{}^{\bar{l}} \quad \text{(A.4)}$$

Almost complex structure on a manifold is a (1,1)- tensor field J with the property $JJ = -\mathbf{1}$, i.e. $J^{i}{}_{j}J^{j}{}_{k} = -\delta^{i}_{k}$

A Kähler form, $F = Jg$, is defined by an almost complex structure on a Riemann manifold: $F_{ji} = -J_{ij} = J_{ji} = J_{j}{}^{k}g_{ki}$

We define barred indices:

$$A^{\bar{k}} := J^{k}{}_{k'}A^{k'} \quad , \quad A_{\bar{k}} := A_{k'}J^{k'}{}_{k} = A^{k'}J_{k'k} = g_{\bar{k}k'}A^{k'} = -g_{kj}J^{j}{}_{k'}A^{k'} = -g_{kj}A^{\bar{j}} \quad \text{(A.5)}$$

(This convention leads to a minus sign when raising or lowering barred indices with the metric tensor.)

We can define a sequence of stronger and more restrictive conditions on almost complex manifolds. Here we use the notation $[D_X, J]Y := D_X(JY) - JD_X(Y) = D_X(J)Y$, the commutator of linear operators acting on vector fields.

- Almost complex structure: $J^{i}{}_{j}J^{j}{}_{k} = -\delta^{i}_{j}$

- Almost Hermitian $\mathfrak{AH}$: $J_{ik} = g_{ij}J^{j}{}_{k} = J_{ki} = g_{kj}J^{j}{}_{i}$

- Semi-Kähler $\mathfrak{SK}$: $D_jJ^{j}{}_{k} = 0$ or $\delta F := *\,\mathrm{d} * F = 0$.

- Quasi-Kähler $\mathfrak{QK}$: $[D_{JX}, J] = -J[D_X, J]$, i.e. $D_iJ^{j}{}_{k'}J^{k'}{}_{k} \;=\; J^{i'}{}_{i}D_{i'}J^{j}{}_{k}$ or $D_iJ^{j}{}_{\bar{k}} = D_{\bar{i}}J^{j}{}_{k} = -D_iJ^{\bar{j}}{}_{k}$ or $D_iJ^{j}{}_{k} = -D_{\bar{i}}J^{j}{}_{\bar{k}} = -J^{i'}{}_{i}J^{k'}{}_{k}D_iJ^{j}{}_{k'}$ (the last equation clearly implies the Semi-Kähler condition) , or

$$\bar{M}_{ik}{}^{j} := -D_iJ^{j}{}_{\bar{k}} + D_{\bar{i}}J^{j}{}_{k} = -D_iJ^{j}{}_{k'}J^{k'}{}_{k} + J^{i'}{}_{i}D_{i'}J^{j}{}_{k} = D_iJ^{\bar{j}}{}_{k} + D_{\bar{i}}J^{j}{}_{k} = D_iJ^{j'}{}_{k'}J^{j}{}_{j'} + J^{i'}{}_{i}D_{i'}J^{j}{}_{k} = 0 \quad \text{(A.6)}$$

or equivalently,

$$\bar{N}_{ik}{}^{j} := 2\bar{M}_{[ik]}^{\;j} = 2(-D_{[i}J^{j}{}_{\bar{k}]} + D_{[\bar{i}}J^{j}{}_{k]}) = (-D_iJ^{j}{}_{k'}J^{k'}{}_{k} + D_kJ^{j}{}_{k'}J^{k'}{}_{i} + J^{i'}{}_{i}D_{i'}J^{j}{}_{k} - J^{i'}{}_{k}D_{i'}J^{j}{}_{i}) = 0 \quad \text{(A.7)}$$

this is equivalent to

$$\overset{-}{N}_{ij}{}^{l} \;=\; 2(\partial_m J_{[j}^{\;l} J_{i]}^{\;m} + \partial_{[i} J_{j]}^{\;m} J_m^{\;l}\;) - 4J_{[i}^{\;r}\,\Gamma_{j]r}{}^{m} J_m{}^{l} = 0 \tag{A.8}$$

where $\overset{-}{N}_{ik}{}^{j}$ is the conjugate Nijenhuis tensor.

Note that the Christoffel symbols do not cancel out.

For a Quasi-Kähler manifold the Nijenhuis tensor simplifies:

$$\overset{+}{N}_{ik}{}^{j} = 4D_{[i}J^{j}{}_{\bar{k}]} = 4D_{[\bar{i}}J^{j}{}_{k]}) = 2(D_iJ^{j}{}_{k'}J^{k'}{}_{k} - D_kJ^{j}{}_{k'}J^{k'}{}_{i}) = 2(J^{i'}{}_{i}D_{i'}J^{j}{}_{k} - J^{i'}{}_{k}D_{i'}J^{j}{}_{i}) \tag{A.9}$$

We also have the Riemann tensor identity: $[D_{\overset{+}{N}(X,Y)}, J] = [\mathfrak{R}_{XY} - \mathfrak{R}_{JXJY}, J] - J[\mathfrak{R}_{JXY} - \mathfrak{R}_{JYX}, J]$

where $\mathfrak{R}$ is the Riemann curvature tensor: $\langle \mathfrak{R}_{XY}(W), \omega\rangle = \mathfrak{R}_{ijk}{}^{l} X^iY^jW^k\omega_l$.

- Almost-Kähler $\mathfrak{AK}$: $\mathrm{d}J = D_{[i}J_{jk]} = 0$, i.e. the Kähler form is symplectic.

- Kahler $\mathfrak{K}$: $D_iJ_{jk} = 0$

- Instead of Almost-Kähler, a Quasi-Kähler manifold can be Nearly-Kähler: $\mathfrak{NK}$: $D_{(i}J_{j)k} = 0$

- A Nearly-Kähler manifold which is also Almost-Kähler is Kähler.

There is also the following sequence of almost Hermitian manifolds with integrable complex structures:

- Hermitian. Integrable complex structure, $\mathfrak{H}$: $D_iJ^{j}{}_{k'}J^{k'}{}_{k} = -J^{i'}{}_{i}D_{i'}J^{j}{}_{k}$ or $D_iJ^{j}{}_{\bar{k}} = -D_{\bar{i}}J^{j}{}_{k} = -D_iJ^{\bar{j}}{}_{k}$ or $D_iJ^{j}{}_{k} = D_{\bar{i}}J^{j}{}_{\bar{k}} = J^{i'}{}_{i}J^{k'}{}_{k}D_{i'}J^{j}{}_{k'}$, or

$$\overset{+}{M}_{ik}{}^{j} := D_iJ^{j}{}_{\bar{k}} + D_{\bar{i}}J^{j}{}_{k} = D_iJ^{j}{}_{k'}J^{k'}{}_{k} + J^{i'}{}_{i}D_{i'}J^{j}{}_{k} = -D_iJ^{\bar{j}}{}_{k} + D_{\bar{i}}J^{j}{}_{k} = -D_iJ^{j'}{}_{k'}J^{j}{}_{j'} + J^{i'}{}_{i}D_{i'}J^{j}{}_{k} = 0 \tag{A.10}$$

or equivalently,

$$\overset{+}{N}_{ik}{}^{j} = 2\overset{+}{M}_{[ik]}{}^{j} = 2(D_{[i}J^{j}{}_{\bar{k}]} + D_{[\bar{i}}J^{j}{}_{k]}) = (D_iJ^{j}{}_{k'}J^{k'}{}_{k} - D_kJ^{j}{}_{k'}J^{k'}{}_{i} + J^{i'}{}_{i}D_{i'}J^{j}{}_{k} - J^{i'}{}_{k}D_{i'}J^{j}{}_{i}) = 0 \tag{A.11}$$

$$\overset{+}{N}_{ij}{}^{l} \;:=\; 2(\partial_m J_{[j}^{\;l} J_{i]}^{\;m} - \partial_{[i} J_{j]}^{\;m} J_m^{\;l}\;) =\; 2(\partial_{[\bar{i}} J_{j]}^{\;l} \;-\; \partial_{[i} J_{j]}^{\;\bar{l}}\;) \tag{A.12}$$

because the Christoffel symbols in the covariant derivatives cancel out and where $\overset{+}{N}_{ik}{}^{j}$ is the

Nijenhuis tensor. The Nijenhuis tensor is indifferent to the particular derivation D_i that is employed above because the Christoffel symbols in the covariant derivatives cancel out.

For a Hermitian manifold the conjugate Nijenhuis tensor simplifies:

$$\bar{N}_{ik}{}^{j} = -4D_{[i}J^{j}{}_{\bar{k}]} = 4D_{[\bar{i}}J^{j}{}_{k]}) = 2(-D_i J^{j}{}_{k'}J^{k'}{}_{k} + D_k J^{j}{}_{k'}J^{k'}{}_{i}) = 2(J^{i'}{}_{i}D_{i'}J^{j}{}_{k} - J^{i'}{}_{k}D_{i'}J^{j}{}_{i}) \tag{A.13}$$

There is also the Riemann tensor identity: $[\mathfrak{R}_{XY} - \mathfrak{R}_{JXJY}, J] + J[\mathfrak{R}_{JXY} - \mathfrak{R}_{JYX}, J] = 0$

- Hermitian semi-Kähler $\mathfrak{H}$: Hermitian and semi-Kahler.

- Kähler $\mathfrak{K}$: $D_i J_{jk} = 0$. This is the same as Hermitian and Quasi-Kähler.

Appendix B

Brackets

Various possibly useful 'brackets' on multiphase space are considered. The first section defines the multi-Poisson bracket, which is used in this thesis. The second section defines brackets on $(d-1)$-form hamiltonian observables via the multisymplectic form. The third section looks at a related question of the functional form of hamiltonian $d-1$-forms, which is the most important category of 'observable' in multisymplectic mechanics. The fourth section defines various other brackets which appear in the literature on manifolds, such as the Schouten bracket, the Schouten-Nijenhuis bracket, the Frolicher-Nijenhuis bracket, the Schouten-Nijenhuis-Richardson bracket, the Buttin bracket, Liebnitz algebras and Lie algebroids. The fifth section looks at the use of the Schouten bracket with relation to hamiltonain multivector fields.

B.1 The multi-Poisson bracket

We will present four slightly different definitions of the multi-Poisson bracket, which will depend on how the index α is absorbed. In these four definitions we ignore the grassmann odd coordinates.

We use notation where Greek letters η^i and ρ_i^α represent grassmann odd (local Darboux) canonical pairs of coordinates and u^i and p_i^α are grassmann even canonical pairs of coordinates on super-multiphase space. The use of supermanifolds, that is, manifolds with some grassmann odd coordinates, is based on the work of DeWitt [16] and Rogers [9].

The multimomenta p_i^α and ρ_i^α are the fiber coordinates on covariant multiphase space. This is considered as a vector bundle over the configuration bundle C, and is the covariant multiphase space P, which has locally adapted coordinates $\sigma^\alpha, u^i, p_i^\alpha$. The canonical d form on P is $\Theta = du^i \wedge p_i^\alpha d\sigma_\alpha -\ \mathrm{d}\eta^i \wedge \rho_i^\alpha \mathrm{d}\sigma_\alpha$ where $d\sigma_\alpha := \partial_\alpha \neg\, d\sigma^0 \wedge d\sigma^1 \wedge \ldots d\sigma^D = \partial_\alpha \neg\, d^d\sigma$ and the multisymplectic form is $\Omega = -d\Theta = -dp_i^\alpha \wedge du^i \wedge d\sigma_\alpha$.

The multisymplectic form (including odd coordinates) in Darboux coordinates is

$$\Omega \ = \ -\mathrm{d}\ (\mathrm{d}u^i \wedge p_i^\alpha \mathrm{d}\sigma_\alpha +\ \mathrm{d}\eta^i \wedge \rho_i^\alpha \mathrm{d}\sigma_\alpha\ ) \ = \ -\mathrm{d}p_i^\alpha \wedge \mathrm{d}u^i \wedge \mathrm{d}\sigma_\alpha +\ \mathrm{d}\rho_i^\alpha \wedge \mathrm{d}\eta^i \wedge \mathrm{d}\sigma_\alpha \tag{B.1}$$

where, on the multiphase space, (u^i, p_i^α) are grassmann even canonical coordinate pairs, and (η^i, ρ_i^α) are grassmann odd canonical coordinate pairs, and $\sigma^\alpha, \alpha = 0 \ldots d-1$ are the spacetime coordinates.

The multi-Poisson bracket is defined as:

$$\{f, g\}_\alpha := f\overleftarrow{\mathrm{d}}_{v\,v} \llcorner \Pi_\alpha \lrcorner \overrightarrow{\mathrm{d}}_v g = -\mathrm{d}_v f \llcorner \Pi_\alpha \lrcorner \mathrm{d}_v g = f \cdot \Pi_\alpha \cdot g := f \cdot (\frac{\overleftarrow{\partial}}{\partial u^i} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha}) \cdot g - f \cdot (\frac{\overleftarrow{\partial}}{\partial \rho_i^\alpha} \wedge \frac{\overrightarrow{\partial}}{\partial \eta^i}) \cdot g$$

$$= \frac{\partial f}{\partial u^i}\frac{\partial g}{\partial p_i^\alpha} \ - \ \frac{\partial f}{\partial p_i^\alpha}\frac{\partial g}{\partial u^i} \ + (-1)^{(|f|+|\rho_i^\alpha|)|\eta^i|}\frac{\partial f}{\partial \eta^i}\frac{\partial g}{\partial \rho_i^\alpha} - (-1)^{|f||\rho_i^\alpha|}\frac{\partial f}{\partial \rho_i^\alpha}\frac{\partial g}{\partial \eta^i} \tag{B.2}$$

where d_v is the exterior derivative on the fibre over the spacetime base space.

If $|\eta^i|, |\rho_i^\alpha|$ are grassmann odd degree: $|\eta^i| = 1 = |\rho_i^\alpha|$ the preceeding is:

$$\{f, g\}_\alpha = \frac{\partial f}{\partial u^i}\frac{\partial g}{\partial p_i^\alpha} \ - \ \frac{\partial f}{\partial p_i^\alpha}\frac{\partial g}{\partial u^i} \ - (-1)^{(|f|)}(\frac{\partial f}{\partial \eta^i}\frac{\partial g}{\partial \rho_i^\alpha} + \frac{\partial f}{\partial \rho_i^\alpha}\frac{\partial g}{\partial \eta^i}) \tag{B.3}$$

A local operation on space-time unlike the Poisson bracket.

When Π and Ω treated as matrices, with matrix multiplication, we obtain

$$\Pi \cdot \Omega = \frac{\partial}{\partial u^i} \otimes \mathrm{d}u^i + \frac{\partial}{\partial p_i^\alpha} \otimes \mathrm{d}p_i^\alpha - \frac{\partial}{\partial \eta^i} \otimes \mathrm{d}\eta^i - \frac{\partial}{\partial \rho_i^\alpha} \otimes \mathrm{d}\rho_i^\alpha = \ 1^{\mathrm{T}} \tag{B.4}$$

$$\Omega \cdot \Pi = (-1)^d (\ \mathrm{d}u^i \otimes \frac{\partial}{\partial u^i} + \mathrm{d}p_i^\alpha \otimes \frac{\partial}{\partial p_i^\alpha} + \mathrm{d}\eta^i \otimes \frac{\partial}{\partial \eta^i} + \mathrm{d}\rho_i^\alpha \otimes \frac{\partial}{\partial \rho_i^\alpha}\ ) \ = \ (-1)^d\ 1 \tag{B.5}$$

B.1.1 Multi-Poisson bracket definion No. 1

We now ignore the grassmann odd coordinates.

We now define a multivector multi-Poisson bracket with the index α contracted with a base space multivector factor $\partial^\alpha \ := \ \frac{\partial}{\partial x_\alpha}$:

$$\{f, g\} := \{f, g\}_\alpha \, \partial^\alpha = f \cdot \Pi \cdot g \tag{B.6}$$

A multihamiltonian tensor field X_f of degree $\binom{r-d+s}{s}$, corresponding to a form observable f of degree $\binom{r}{0}$ on multiphase space, is defined by $X_f \lrcorner \Omega = \mathrm{d}f$, so we have $X_{u^i} = (-1)^{d-1}\,\partial^\alpha \wedge \frac{\partial}{\partial p_i^\alpha} = \frac{\partial}{\partial p_i^\alpha} \wedge \partial^\alpha$ of degree $\binom{0}{d}$, this gives, corresponding to the 0-form u^i,

$$X_{u^i} \lrcorner \Omega = \mathrm{d}u^i \tag{B.7}$$

and, employing the definition of the multi-bracket above,

$$\{u^i, \cdot\} = \{u^i, \cdot\}_\alpha\, \partial^\alpha = \frac{\partial}{\partial p_i^\alpha} \wedge \partial^\alpha = X_{u^i} \tag{B.8}$$

Similarly, for $p_i := p_i^\alpha \mathrm{d}\sigma_\alpha$:

$$X_{p_i} = \{p_i, \cdot\} = \{p_i^\alpha \mathrm{d}\sigma_\alpha \llcorner, \cdot\}_\beta\, \partial^\beta = (-1)^d\, \mathrm{d}\sigma_\alpha \otimes \partial^\alpha \wedge \frac{\partial}{\partial u^i} \tag{B.9}$$

then $X_{p_i} \lrcorner \Omega = \mathrm{d}p_i = \mathrm{d}p_i^\alpha \wedge \mathrm{d}\sigma_\alpha$.

B.1.2 Multi-Poisson bracket definion No. 2

We now consider a slighly different definition of the multibracket where the base space (space-time) multivector is wedged between $\frac{\partial}{\partial u^i}$ and $\frac{\partial}{\partial p_i^\alpha}$:

$$\{f, g\} := f\overleftarrow{\mathrm{d}}_v \llcorner \Pi \lrcorner \overrightarrow{\mathrm{d}}_v g = -\mathrm{d}_v f \llcorner \Pi \lrcorner \mathrm{d}_v g = f \cdot \Pi \cdot g := f \cdot (\frac{\overleftarrow{\partial}}{\partial u^i} \wedge \frac{\partial}{\partial x_\alpha} \wedge \frac{\overrightarrow{\partial}}{\partial p_i^\alpha}) \cdot g - f \cdot (\frac{\overleftarrow{\partial}}{\partial \rho_i^\alpha} \wedge \frac{\partial}{\partial x_\alpha} \wedge \frac{\overrightarrow{\partial}}{\partial \eta^i}) \cdot g$$

$$-f \cdot (\frac{\overleftarrow{\partial}}{\partial p_i^\alpha} \wedge \frac{\partial}{\partial x_\alpha} \wedge \frac{\overrightarrow{\partial}}{\partial u^i}) \cdot g \; + (-1)^{|\rho||\eta|}\; f \cdot (\frac{\overleftarrow{\partial}}{\partial \eta^i} \wedge \frac{\partial}{\partial x_\alpha} \wedge \frac{\overrightarrow{\partial}}{\partial \rho_i^\alpha}) \cdot g \tag{B.10}$$

Now a slight change of notation: $\frac{\partial}{\partial x_\alpha} \;=:\; \partial^\alpha$ is introduced.

With this definition of the multibracket we will now examine the multihamiltonian tensors corresponding to the 0-form u^i and the $d-1$-form p_i.

$$X_{u^i} = \{u^i, \cdot\} = (-1)^{d-1}\frac{\partial}{\partial p_i^\alpha} \wedge \partial^\alpha = \partial^\alpha \wedge \frac{\partial}{\partial p_i^\alpha} \tag{B.11}$$

then $X_{u^i} \lrcorner \Omega = (-1)^{d-1}\,\mathrm{d}u^i\;\delta_\alpha^\alpha$ and $\Omega \llcorner X_{u^i} = (-1)^{d-1}\,\mathrm{d}u^i\;\delta_\alpha^\alpha$.

$$X^T_{u^i} = \{\cdot, u^i\} = -\frac{\partial}{\partial p_i^\alpha} \wedge \partial^\alpha = -(-1)^{d-1}\partial^\alpha \wedge \frac{\partial}{\partial p_i^\alpha} \tag{B.12}$$

then $X^T_{u^i} \lrcorner \Omega = -\,\mathrm{d}u^i\;\delta_\alpha^\alpha$ and $\Omega \llcorner X^T_{u^i} = -\,\mathrm{d}u^i\;\delta_\alpha^\alpha$.

Similarly, for $p_i := p_i^\alpha \mathrm{d}\sigma_\alpha$:

$$X_{p_i} = \{p_i, \cdot\} = \{p_i^\alpha \mathrm{d}\sigma_\alpha, \cdot\} \;=\; (-1)\,\mathrm{d}\sigma_\alpha \otimes \partial^\alpha \wedge \frac{\partial}{\partial u^i} \tag{B.13}$$

then $X_{p_i} \wedge \lrcorner \Omega = \mathrm{d}p_i = \mathrm{d}p_i^\alpha \wedge \mathrm{d}\sigma_\alpha$.

or

$$X_{p_i} = \{p_i, \cdot\} = \{p_i^\alpha \mathrm{d}\sigma_\alpha, \cdot\} \ = \ -\mathrm{d}\sigma_\alpha \llcorner \partial^\alpha \wedge \frac{\partial}{\partial u^i} = \delta_\alpha^\alpha \frac{\partial}{\partial u^i} \tag{B.14}$$

then $X_{p_i} \wedge \lrcorner \Omega = \mathrm{d}p_i = \mathrm{d}p_i^\alpha \wedge \mathrm{d}\sigma_\alpha$,

and also $\{u^k, p_l\} = \delta_l^k \delta_\alpha^\alpha$.

Poisson bracket with DDW Hamiltonian $\tilde{H} = H\mathrm{d}^d x$:

$$\{p_i, \tilde{H}\} = \{p_i^\alpha \mathrm{d}\sigma_\alpha, H\mathrm{d}^d x\} \ = \ -\mathrm{d}\sigma_\alpha \llcorner \partial^\alpha \wedge \frac{\partial}{\partial u^i} \cdot H \ \mathrm{d}^d x = \delta_\alpha^\alpha \frac{\partial H}{\partial u^i} \ \mathrm{d}^d x \tag{B.15}$$

This would be $\approx$ (d) ($\partial_\alpha p_i^\alpha$) $\mathrm{d}^d x$ $=$ $(d)\mathrm{d}p_i$, the divergence DDW eqution of motion, where $\mathrm{d}p_i$ is $\mathrm{d}(p_i^\alpha(\sigma)\mathrm{d}\sigma_\alpha)$, the exterior derivative of a form on spacetime, namely that of a multimomentum field configuration corresponding to a solution of the equations of motion. The factor of d is inserted because of the $\delta_\alpha^\alpha = d$ produced in the Poisson bracket calculation.

Similarly,

$$(\ \partial_\alpha u^i\ )\ \mathrm{d}x^\alpha \ = \ \mathrm{d}u^i \ \approx \ \{u^i, \tilde{H}\} = \{u^i, H\mathrm{d}^d x\} \ = \ \partial^\alpha \wedge \frac{\partial}{\partial p_i^\alpha} \cdot H \ \ \lrcorner\, \mathrm{d}^d x = (-1)^{d-1-\alpha} \ \frac{\partial H}{\partial p_i^\alpha} \ \mathrm{d}x^\alpha \tag{B.16}$$

$$(\ \partial_\alpha u^i\ )\ \mathrm{d}x^\alpha \ = \ \mathrm{d}u^i \ \approx \ \{\tilde{H}, u^i\} = \{H\mathrm{d}^d x, u^i\} \ = \ \partial^\alpha \wedge \frac{\partial}{\partial p_i^\alpha} \cdot H \ \ \lrcorner\, \mathrm{d}^d x = (-1)^{d-1-\alpha} \ \frac{\partial H}{\partial p_i^\alpha} \ \mathrm{d}x^\alpha \tag{B.17}$$

B.1.3 Multi-Poisson bracket definion No. 3

Multivector multipoisson bracket as a 1-form:

$$\{f, g\} := \mathrm{d}x^\alpha \ \{f, g\}_\alpha \tag{B.18}$$

With this definition of the multibracket we will now examine the multihamiltonian tensors corresponding to the 0-form u^i and the $d-1$-form p_i.

$$X_{u^i} = \{u^i, \cdot\} = \mathrm{d}x^\alpha \otimes \frac{\partial}{\partial p_i^\alpha} \tag{B.19}$$

then $X_{u^i} \wedge \lrcorner \Omega = -\mathrm{d}u^i \wedge \mathrm{d}^d x = -\mathrm{d}(u^i \wedge \mathrm{d}^d x)$.

Similarly, for $p_i := p_i^\alpha \mathrm{d}x_\alpha$:

$$X_{p_i} = \{p_i, \cdot\} = -\mathrm{d}^d x \otimes \frac{\partial}{\partial u^i} \tag{B.20}$$

then $X_{p_i} \lrcorner \Omega = \mathrm{d}^d x \otimes \mathrm{d}p_i^\alpha \wedge \mathrm{d}x_\alpha = \mathrm{d}^d x \otimes \mathrm{d}(p_i^\alpha \wedge \mathrm{d}x_\alpha)$.

B.1.4 Multi-Poisson bracket definion No. 4

Multi-Poisson bracket as a 1-form placed on right of bracket:

$$\{f, g\} := \{f, g\}_\alpha \, \mathrm{d}x^\alpha \tag{B.21}$$

With this definition of the multibracket we will now examine the multihamiltonian tensors corresponding to the 0-form u^i and the $d-1$-form p_i.

$$X_{u^i} = \{u^i, \cdot\} = \frac{\partial}{\partial p_i^\alpha} \otimes \mathrm{d}x^\alpha \tag{B.22}$$

then $X_{u^i} \wedge \lrcorner\, \Omega = \mathrm{d}u^i \wedge \mathrm{d}^d x = \mathrm{d}(u^i \wedge \mathrm{d}^d x)$.

Similarly, for $p_i := p_i^\alpha \mathrm{d}x_\alpha$:

$$X_{p_i} = \{p_i, \cdot\} = (-1)^{(d-1)} \frac{\partial}{\partial u^i} \otimes \mathrm{d}^d x \tag{B.23}$$

then $X_{p_i} \lrcorner\, \Omega \;=\; -(-1)^{d(d-1)} \, \mathrm{d}p_i^\alpha \wedge \mathrm{d}x_\alpha \otimes \mathrm{d}^d x \;=\; -(-1)^{d(d-1)} \, \mathrm{d}(p_i^\alpha \wedge \mathrm{d}x_\alpha) \otimes \mathrm{d}^d x$.

B.2 Bracket on $(d-1)$-form hamiltonian observables

Immediately generalizing the definition of a bracket on a symplectic manifold to multisymplectic manifolds one can define a "bracket" [18] on $(d-1)$-form hamiltonian observables in two ways: a "semi-bracket": $\{F, G\}_s := X_F \lrcorner\, X_G \lrcorner\, \Omega$, where Ω is the $d+1$-multisymplectic form, and X_F is the hamiltonian vector field associated with the hamiltonian $d-1$-form F : $\mathrm{d}F := X_F \lrcorner\, \Omega$. This bracket is anti-symmetric, but only satisfies the Jacobi identity up to an exact form:

$$J_{F,G,H} := \{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = d(X_F \lrcorner\, X_G \lrcorner\, X_H \lrcorner\, \Omega) \tag{B.24}$$

and a "hemi-bracket" $\{F, G\}_h := \mathcal{L}_{X_F} G = \{F, G\}_s + d(X_F \lrcorner\, G)$ which is satisfies the Jacobi identity but is only anti-symmetric up to an exact form:

$$J_{F,G,H} := \{F, \{G, H\}\} + \{H, \{F, G\}\} + \{G, \{H, F\}\} = d(X_F \lrcorner\, X_G \lrcorner\, X_H \lrcorner\, \Omega) \tag{B.25}$$

For both these brackets the result is a hamiltonian (d-1)-form, and they are the same up to an exact form :

$$[X_F, X_G] = X_{\{F,G\}_h} = X_{\{F,G\}_s} \tag{B.26}$$

Therefore the hamiltonian (d-1)-forms mod closed or exact (d-1)-forms with $\{,\}_s$ are Lie algebras. Similarly one can define a bracket between two Hamiltonian forms F, G of form degree p and q respectively: $\{F, G\}_s := (-1)^{g_F} X_F \lrcorner X_G \lrcorner \Omega$, which is a hamiltonian (p+q-d+1)-form. The set of hamiltonian forms mod closed forms form a graded Lie algebra where the grading degree is $d-p-1$ for a p-form. The same definition as B.26 can be given for generalized hamiltonian forms, but in this case the bracket is non-commutative, yet satisfies the left graded Loday identity and the right graded Leibniz rule [83]. This Poisson-Leibniz algebra of observables has the required structure to construct a BRST procedure [84].

B.3 Functional form of hamiltonian $d-1$-forms

The structure of Ω strongly constrains the functional form of the components of the vector field X_F and the $d-1$ form F. We investigate this in multiphase space by explicitly writing out the most general expansion of $X_F \lrcorner \Omega = \mathrm{d}F$ in a coordinate basis and then identifying the components on each side of the equation. This calculation of the multi-Poisson bracket may possibly serve as a definition of the brackets for the more general case of $d-1$-form observables which are not hamiltonian.

The left hand side is

$$X_F \lrcorner \Omega \;=\; U^i(u^j, p_j^\beta, x)\frac{\partial}{\partial u^i} + P_i^\alpha(u^j, p_j^\beta, x)\frac{\partial}{\partial p_i^\alpha} + X^\alpha(u^j, p_j^\beta, x)\frac{\partial}{\partial x^\alpha} \quad \lrcorner \;\; \mathrm{d}u^i \wedge \mathrm{d}p_i^\alpha \wedge \mathrm{d}^{d-1}x_\alpha$$

$$= \; U^i(u^j, p_j^\beta, x)\, \mathrm{d}p_i^\alpha \wedge \mathrm{d}^{d-1}x_\alpha - P_i^\alpha(u^j, p_j^\beta, x)\, \mathrm{d}u^i \wedge \mathrm{d}^{d-1}x_\alpha + X^\alpha(u^j, p_j^\beta, x)\, \mathrm{d}u^i \wedge \mathrm{d}p_i^\beta \wedge \mathrm{d}^{d-2}x_{\alpha\beta} \tag{B.27}$$

The right hand side is

$$\mathrm{d}F = \mathrm{d}\,[\; F^\alpha(u^j, p_j^\beta)\, \mathrm{d}^{d-1}x_\alpha + F_i^{\alpha\beta}(p_j^\beta, x)\, \mathrm{d}u^i \wedge \mathrm{d}^{d-2}x_{\alpha\beta} + F^{i\alpha}(u^j, x)\, \mathrm{d}p_i^\beta \wedge \mathrm{d}^{d-2}x_{\alpha\beta}$$

$$+\; F^{\alpha\beta}(x)\, \mathrm{d}u^i \wedge \mathrm{d}p_i^\gamma \wedge \mathrm{d}^{d-3}x_{\alpha\beta\gamma}\;]$$

$$= \; \frac{\partial F^\alpha}{\partial u^i}\, \mathrm{d}u^i \wedge \mathrm{d}^{d-1}x_\alpha + \; \frac{\partial F^\alpha}{\partial p_i^\beta}\, \mathrm{d}p_i^\beta \wedge \mathrm{d}^{d-1}x_\alpha +$$

$$\frac{\partial F_i^{\alpha\beta}}{\partial p_k^\gamma}\;\; \mathrm{d}p_k^\gamma \wedge \mathrm{d}u^i \wedge \mathrm{d}^{d-2}x_{\alpha\beta} + \frac{\partial F_i^{\alpha\beta}}{\partial x^\gamma}\; \mathrm{d}x^\gamma \wedge \mathrm{d}u^i \wedge \mathrm{d}^{d-2}x_{\alpha\beta} + \frac{\partial F^{i\alpha}}{\partial u^k}\; \mathrm{d}u^k \wedge \mathrm{d}p_i^\beta \wedge \mathrm{d}^{d-2}x_{\alpha\beta} +$$

$$\frac{\partial F^{i\alpha}}{\partial x^\gamma}\; \mathrm{d}x^\gamma \wedge \mathrm{d}p_i^\beta \wedge \mathrm{d}^{d-2}x_{\alpha\beta} + \frac{\partial F^{\alpha\beta}}{\partial x^\delta}\; \mathrm{d}x^\delta \wedge \mathrm{d}u^i \wedge \mathrm{d}p_i^\gamma \wedge \mathrm{d}^{d-3}x_{\alpha\beta\gamma}$$

$$= \;(\; \frac{\partial F^\alpha}{\partial u^i} + 2\frac{\partial F_i^{\alpha\beta}}{\partial x^\beta}\;)\; \mathrm{d}u^i \wedge \mathrm{d}^{d-1}x_\alpha + \;(\frac{\partial F^\alpha}{\partial p_i^\beta} + \frac{\partial F^{i\alpha}}{\partial x^\beta})\; \mathrm{d}p_i^\beta \wedge \mathrm{d}^{d-1}x_\alpha - \frac{\partial F^{i\gamma}}{\partial x^\gamma}\; \mathrm{d}p_i^\alpha \wedge \mathrm{d}^{d-1}x_\alpha$$

$$-\frac{\partial F_i^{\alpha\beta}}{\partial p_k^\gamma}\;\; \mathrm{d}u^i \wedge \mathrm{d}p_k^\gamma \wedge \mathrm{d}^{d-2}x_{\alpha\beta} + \frac{\partial F^{i\alpha}}{\partial u^k}\; \mathrm{d}u^k \wedge \mathrm{d}p_i^\beta \wedge \mathrm{d}^{d-2}x_{\alpha\beta} +$$

$$+2\frac{\partial F^{[\alpha\beta]}}{\partial x^\alpha}\ \mathrm{d}u^i\wedge \mathrm{d}p_i^\gamma\wedge \mathrm{d}^{d-2}x_{\beta\gamma}+(\frac{\partial F^{\alpha\beta}}{\partial x^\gamma})\mathrm{d}u^i\wedge \mathrm{d}p_i^\gamma\wedge \mathrm{d}^{d-2}x_{\alpha\beta} \tag{B.28}$$

Some of the components on the right hand side are not present on the left hand side so the corresponding coefficients are required to be zero:

$$(\frac{\partial F^\alpha}{\partial p_i^\beta}(u,p)+\frac{\partial F^{i\alpha}}{\partial x^\beta}(u,x)) \;=\; 0 \text{ if } \alpha\neq\beta \tag{B.29}$$

$$(\frac{\partial F^{\dot\beta}}{\partial p_i^{\dot\beta}}(u,p)+\frac{\partial F^{i\dot\beta}}{\partial x^{\dot\beta}}(u,x)) \;=\; K_i \text{ independent of the value of } \dot\beta.\text{ (no summation of the } \dot\beta\text{'s).} \tag{B.30}$$

$$\frac{\partial F_i^{\alpha\beta}}{\partial p_k^\gamma}(p,x) \;=\; 0 \text{ if } \gamma\neq\alpha \text{ and } \gamma\neq\beta \text{ and } i\neq k \tag{B.31}$$

$$-\frac{\partial F_{\dot k}^{\alpha\beta}}{\partial p_{\dot k}^\gamma}(p,x)+\frac{\partial F^{\alpha\beta}}{\partial x^\gamma}(x) \;=0 \text{ if } \gamma\neq\alpha \text{ and } \gamma\neq\beta \text{ (no summation of the } \dot k\text{'s)} \tag{B.32}$$

$$-\frac{\partial F_i^{\alpha\dot\beta}}{\partial p_k^{\dot\beta}}(p,x)+\frac{\partial F^{k\alpha}}{\partial u^i}(u,x) \;=\; 0 \text{ (no summation of the } \dot\beta\text{'s) if } i\neq k \tag{B.33}$$

$$-\frac{\partial F_{\dot k}^{\alpha\dot\beta}}{\partial p_{\dot k}^{\dot\beta}}(p,x)+\frac{\partial F^{\dot k\alpha}}{\partial u^{\dot k}}(u,x) \;=\; L_{\dot k} \text{ independent of the value of } \dot\beta.\text{ (no summation of the } \dot\beta\text{'s or the } \dot k\text{'s)} \tag{B.34}$$

The first of condition implies that, if $\beta\neq\alpha$, F^α is linear in the p_i^β's and that $F^{i\alpha}$ is linear in the x^β's .

The third condition implies that $F_i^{\alpha\beta}$ is independent of p_k^γ if $\gamma\neq\alpha$ and $\gamma\neq\beta$ and $i\neq k$.

The fourth condition implies that, if $\gamma\neq\alpha$ and $\gamma\neq\beta$, $F_{\dot k}^{\alpha\beta}$ is linear in $p_{\dot k}^\gamma$'s and that $F^{i\alpha}$ is linear in the x's.

The fifth condition implies that, if $i\neq k$, $F_i^{\alpha\beta}$ is linear in p_k^α's and that $F^{k\alpha}$ is linear in the x^i's.

If these conditions above hold then

$$\mathrm{d}F=[\,(\frac{\partial F^\alpha}{\partial u^i}+2\frac{\partial F_i^{\alpha\beta}}{\partial x^\beta})\ \mathrm{d}u^i\wedge \mathrm{d}^{d-1}x_\alpha+\ (\frac{1}{d}(\frac{\partial F^\gamma}{\partial p_i^\gamma}+\frac{\partial F^{i\gamma}}{\partial x^\gamma})-\frac{\partial F^{i\gamma}}{\partial x^\gamma})\ \mathrm{d}p_i^\alpha\wedge \mathrm{d}^{d-1}x_\alpha$$
$$+\,(-\frac{1}{dK}\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{1}{dK}\frac{\partial F^{k\beta}}{\partial u^k}+\frac{2}{d}\frac{\partial F^{[\alpha\beta]}}{\partial x^\alpha}+2\frac{\partial F^{[\alpha\beta]}}{\partial x^\alpha})\ \mathrm{d}u^i\wedge \mathrm{d}p_i^\gamma\wedge \mathrm{d}^{d-2}x_{\beta\gamma}\,] \tag{B.35}$$

If $F_k^{\alpha\beta}$, $F^{\alpha\beta}$ are not explicitly functions of x, this simplifies to

$$\mathrm{d}F=\frac{\partial F^\alpha}{\partial u^i}\ \mathrm{d}u^i\wedge \mathrm{d}^{d-1}x_\alpha+\frac{1}{d}\frac{\partial F^\gamma}{\partial p_i^\gamma}\ \mathrm{d}p_i^\alpha\wedge \mathrm{d}^{d-1}x_\alpha+\frac{1}{dK}(-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{\partial F^{k\beta}}{\partial u^k})\ \mathrm{d}u^i\wedge \mathrm{d}p_i^\gamma\wedge \mathrm{d}^{d-2}x_{\beta\gamma} \tag{B.36}$$

The corresponding hamiltonian vector field is

$$X_F \;=\; U^i\frac{\partial}{\partial u^i}+P_i^\alpha\frac{\partial}{\partial p_i^\alpha}+X^\beta\frac{\partial}{\partial x^\beta} \;=\; (\frac{1}{d}(\;\frac{\partial F^\gamma}{\partial p_i^\gamma}+\frac{\partial F^{i\gamma}}{\partial x^\gamma}\;)-\frac{\partial F^{i\gamma}}{\partial x^\gamma}\;)\frac{\partial}{\partial u^i}\;-\;(\;\frac{\partial F^\alpha}{\partial u^i}+2\frac{\partial F_i^{\alpha\beta}}{\partial x^\beta}\;)\;\frac{\partial}{\partial p_i^\alpha}$$

$$+\;(\;-\frac{1}{dK}\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{1}{dK}\frac{\partial F^{k\beta}}{\partial u^k}+\frac{2}{d}\frac{\partial F^{[\alpha\beta]}}{\partial x^\alpha}+2\frac{\partial F^{[\alpha\beta]}}{\partial x^\alpha}\;)\;\frac{\partial}{\partial x^\beta} \tag{B.37}$$

We now calculate the Multi-Poisson Brackets

$$\{G,F\} = -\{F,G\} = X_G \lrcorner \mathrm{d}F = X_G \lrcorner X_F \lrcorner \Omega \tag{B.38}$$

The Multi-Poisson Brackets expanding F and G, in the case that F is not explicitly a function of x,

$$\{G,F\} = X_G \lrcorner \mathrm{d}F =$$

$$[\;U^i\frac{\partial}{\partial u^i}+P_i^\alpha\frac{\partial}{\partial p_i^\alpha}+X^\alpha\frac{\partial}{\partial x^\alpha}\;]\;\lrcorner\;[\;(\frac{\partial F^\alpha}{\partial u^i}\;\mathrm{d}u^i+\frac{1}{d}\frac{\partial F^\gamma}{\partial p_i^\gamma}\;\mathrm{d}p_i^\alpha)\wedge\mathrm{d}^{d-1}x_\alpha\;+$$

$$\frac{1}{dK}(\;-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{\partial F^{k\beta}}{\partial u^k}\;)\;\mathrm{d}u^i\wedge\mathrm{d}p_i^\gamma\wedge\mathrm{d}^{d-2}x_{\beta\gamma}\;]$$

$$=\;\;[\;\;(U^i\frac{\partial F^\alpha}{\partial u^i}+\frac{1}{d}P_i^\alpha\frac{\partial F^\gamma}{\partial p_i^\gamma})\mathrm{d}^{d-1}x_\alpha\;\;+\;\;X^\kappa(\frac{\partial F^\alpha}{\partial u^i}\;\mathrm{d}u^i+\frac{1}{d}\frac{\partial F^\gamma}{\partial p_i^\gamma}\;\mathrm{d}p_i^\alpha)\wedge\mathrm{d}^{d-1}x_{\kappa\alpha}\;+$$

$$\frac{1}{dK}[U^i(\;-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{\partial F^{k\beta}}{\partial u^k}\;)\mathrm{d}p_i^\gamma\wedge\mathrm{d}^{d-2}x_{\beta\gamma}\;\;-\;\;P_i^\gamma(\;-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{\partial F^{k\beta}}{\partial u^k}\;)\;\mathrm{d}u^i\wedge\mathrm{d}^{d-2}x_{\beta\gamma}\;+$$

$$X^\kappa(\;-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{\partial F^{k\beta}}{\partial u^k}\;)\;\mathrm{d}u^i\wedge\mathrm{d}p_i^\gamma\wedge\mathrm{d}^{d-3}x_{\kappa\beta\gamma}\;\;] \tag{B.39}$$

Substituting for the components $U^i, P_i^\alpha, X^\kappa$ of X_G (where G may be explicitly a function of x unlike F), we have

$$X_G \lrcorner \mathrm{d}F =\;[\;((\frac{1}{d}(\;\frac{\partial G^\gamma}{\partial p_i^\gamma}+\frac{\partial G^{i\gamma}}{\partial x^\gamma}\;)-\frac{\partial G^{i\gamma}}{\partial x^\gamma}\;)\frac{\partial F^\alpha}{\partial u^i}-\frac{1}{d}(\;\frac{\partial G^\alpha}{\partial u^i}+2\frac{\partial G_i^{\alpha\beta}}{\partial x^\beta}\;)\frac{\partial F^\gamma}{\partial p_i^\gamma})\mathrm{d}^{d-1}x_\alpha\;+$$

$$(\;-\frac{1}{dK}\frac{\partial G_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{1}{dK}\frac{\partial G^{k\beta}}{\partial u^k}+\frac{2}{d}\frac{\partial G^{[\alpha\beta]}}{\partial x^\alpha}+2\frac{\partial G^{[\alpha\beta]}}{\partial x^\alpha}\;)(\frac{\partial F^\alpha}{\partial u^i}\;\mathrm{d}u^i+\frac{1}{d}\frac{\partial F^\gamma}{\partial p_i^\gamma}\;\mathrm{d}p_i^\alpha)\wedge\mathrm{d}^{d-1}x_{\beta\alpha}\;+$$

$$\frac{1}{dK}[(\frac{1}{d}(\;\frac{\partial G^\gamma}{\partial p_i^\gamma}+\frac{\partial G^{i\gamma}}{\partial x^\gamma}\;)-\frac{\partial G^{i\gamma}}{\partial x^\gamma}\;)(\;-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{\partial F^{k\beta}}{\partial u^k}\;)\mathrm{d}p_i^\gamma\wedge\mathrm{d}^{d-2}x_{\beta\gamma}\;+$$

$$(\;\frac{\partial G^\gamma}{\partial u^i}+2\frac{\partial G_i^{\gamma\beta}}{\partial x^\beta}\;)(\;-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{\partial F^{k\beta}}{\partial u^k}\;)\;\mathrm{d}u^i\wedge\mathrm{d}^{d-2}x_{\beta\gamma}\;+$$

$$(\;-\frac{1}{dK}\frac{\partial G_k^{\alpha\kappa}}{\partial p_k^\alpha}+\frac{1}{dK}\frac{\partial G^{k\kappa}}{\partial u^k}+\frac{2}{d}\frac{\partial G^{[\alpha\kappa]}}{\partial x^\alpha}+2\frac{\partial G^{[\alpha\kappa]}}{\partial x^\alpha}\;)(\;-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha}+\frac{\partial F^{k\beta}}{\partial u^k}\;)\;\mathrm{d}u^i\wedge\mathrm{d}p_i^\gamma\wedge\mathrm{d}^{d-3}x_{\kappa\beta\gamma}\;\;] \tag{B.40}$$

If we restrict to the case where neither G or F are explicitly functions of x, we have

$$X_G \lrcorner \mathrm{d}F =\;[\;\;\frac{1}{d}(\;\frac{\partial G^\gamma}{\partial p_i^\gamma}\;\frac{\partial F^\alpha}{\partial u^i}\;\;-\;\;\frac{\partial G^\alpha}{\partial u^i}\;\frac{\partial F^\gamma}{\partial p_i^\gamma})\mathrm{d}^{d-1}x_\alpha\;+$$

$$\frac{1}{dK}\big(-\frac{\partial G_k^{\alpha\beta}}{\partial p_k^\alpha} + \frac{\partial G^{k\beta}}{\partial u^k} \big)\big(\frac{\partial F^\alpha}{\partial u^i}\ \mathrm{d}u^i + \frac{1}{d}\frac{\partial F^\gamma}{\partial p_i^\gamma}\ \mathrm{d}p_i^\alpha\big) \wedge \mathrm{d}^{d-1}x_{\beta\alpha} +$$

$$\frac{1}{dK}\frac{1}{d}\ \frac{\partial G^\gamma}{\partial p_i^\gamma}\ \big(-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha} + \frac{\partial F^{k\beta}}{\partial u^k}\ \big)\mathrm{d}p_i^\gamma \wedge \mathrm{d}^{d-2}x_{\beta\gamma} +$$

$$\big(\frac{\partial G^\gamma}{\partial u^i}\ \big)\big(-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha} + \frac{\partial F^{k\beta}}{\partial u^k}\ \big)\ \mathrm{d}u^i \wedge \mathrm{d}^{d-2}x_{\beta\gamma} +$$

$$\frac{1}{dK}\big(-\frac{\partial G_k^{\alpha\kappa}}{\partial p_k^\alpha} + \frac{\partial G^{k\kappa}}{\partial u^k}\ \big)\big(-\frac{\partial F_k^{\alpha\beta}}{\partial p_k^\alpha} + \frac{\partial F^{k\beta}}{\partial u^k}\ \big)\ \mathrm{d}u^i \wedge \mathrm{d}p_i^\gamma \wedge \mathrm{d}^{d-3}x_{\kappa\beta\gamma}\ \Big] \tag{B.41}$$

If we further restrict to the case where $G = G^\alpha(u^j, p_j^\beta)\ \mathrm{d}^{d-1}x_\alpha$ and $F = F^\alpha(u^j, p_j^\beta)\ \mathrm{d}^{d-1}x_\alpha$, we now obtain the more limited multiPoisson bracket (which was examined in previous section):

$$X_G \lrcorner\, \mathrm{d}F = \frac{1}{d}\big(\ \frac{\partial G^\gamma}{\partial p_i^\gamma}\ \frac{\partial F^\alpha}{\partial u^i}\ -\ \frac{\partial G^\alpha}{\partial u^i}\ \frac{\partial F^\gamma}{\partial p_i^\gamma}\big)\mathrm{d}^{d-1}x_\alpha \tag{B.42}$$

B.4 Various other brackets

Certain other types of brackets ([75] [18] [1]) on various algebraic objects have been defined, which may be relevant to dynamics. Some of these are super-brackets in that they may commute for odd elements and obey a super-Jacobi identity. Some of these are briefly introduced in this section.

B.4.1 Algebra of derivations on forms

Part of the research program in multisymplectic dynamics is to find useful sets of "observables" and a Poisson algebra on these "observables" analogous to the Poisson algebra of functions on a symplectic manifold, where these observables may be forms or tensors. To begin with, there are several known graded Lie algebras associated with smooth manifolds which can best be viewed as subalgebras of the graded Lie algebra of derivations, $Der(\Omega(M))$, of differential forms with the graded commutator bracket. These derivations are defined via vector valued forms and act on forms on a manifold [75] [17]: any derivation $D \in Der(\Omega(M))$ can be expressed uniquely as a sum of elements from two subalgebras of derivations: a Lie derivation and a generalized contraction: $D = \mathcal{L}_x + i_y$, for some multivectors x and y: the action of the grade degree $-|y|$ insertion operator i_y on a form ω is contraction with the multivector y: $i_y\omega := y \lrcorner\, \omega$ and the action of the grade degree $|x| - 1$ Lie derivative $\mathcal{L}_x$ is defined as the graded commutator of i_x and the (grade degree 1) exterior derivative d: $\mathcal{L}_x\omega := [\mathrm{d}, i_x]\ \omega = \mathrm{d}(i_x\omega) - (-1)^{|x|}i_x(\mathrm{d}\omega)$. The $\mathcal{L}_x$ form a graded Lie subalgebra of all the derivations D_x such that $[D_x, \mathrm{d}] = 0$.

B.4.2 Schouten bracket

The Schouten bracket of symmetric multi-vector fields on M is naturally isomorphic to the Poisson algebra of functions on T^*M, which are polynomial in the dual tangent space coordinates.

B.4.3 Schouten-Nijenhuis bracket

The Schouten-Nijenhuis bracket is also sometimes called the Schouten bracket.

The set of skew-symmetric multivector fields $\Gamma(\Lambda^\bullet(TM))$, which form a graded algebra under the wedge product, is supercommutative of degree 0, where the grading of a term is the number of vectors factors wedged together in a term: $|X| = |X_1 \wedge \ldots \wedge X_m| = m$

The Schouten-Nijenhuis bracket of skew-symmetric multivector fields is an extension of the Lie bracket of vector fields defined on simple multivectors by:

$$[X,Y] = [X_1 \wedge \ldots \wedge X_m, Y_1 \wedge \ldots \wedge Y_n]$$

$$:= \sum_{i,j} (-1)^{i+j} [X_i, Y_j] \wedge X_1 \wedge \ldots \wedge X_{i-1} \wedge X_{i+1} \ldots \wedge X_m \wedge Y_1 \wedge \ldots Y_{j-1} \wedge Y_{j+1} \wedge \ldots \wedge Y_n \quad \text{(B.43)}$$

where $[X_i, Y_j]$ is the Lie bracket of vectors fields X_i and Y_j. The above is then linearly extended to composite and non-homogenous-degree multivectors. The algebra of multivector fields together with the Schouten-Nijenhuis bracket forms a Gerstenhaber algebra.

A Gerstenhaber algebra $\mathfrak{U}^\bullet$ is an $\mathbb{N}$ graded vector space ($|\mathfrak{U}^m| = m$) with an associative degree 0 supercommutative product $\wedge$, a graded Lie bracket of degree -1, and the following adjoint action Leibnitz rule:

$$[A, B \wedge C] = [A,B] \wedge C + (-1)^{(|A|-1)|B|} B \wedge [A,C] = [A,B] \wedge C - (-1)^{(|C|-1)|B|} [A,C] \wedge B \quad \text{(B.44)}$$

There is a unique Gerstenhaber algebra defined $\Lambda^\bullet \mathfrak{g}$ for any Lie algebra $\mathfrak{g}$, and the Schouten-Nijenhuis bracket is such an extension of the Lie algebra of vector fields on a manifold.

A scalar function is a degree 0 vector field here. For the SN bracket, if one of the factors is a scalar function f: $[f, Y] = -df \llcorner Y$. If both factors scalar functions: $[f, g] := 0$.

The Lie grade degree of the bracket is -1: $|[X,Y]| = |X| + |Y| - 1$ The products are super commuting with different grading $[X,Y] = -(-1)^{(|X|-1)(|Y|-1)}[Y,X]$ as opposed to $X \wedge Y = -(-1)^{(|X|)(|Y|)} Y \wedge X$

This obeys the (graded) Jacobi identity for such a graded Lie algebra:

$$(-1)^{(|X|-1)(|Z|-1)}[X,[Y,Z]]+(-1)^{(|Y|-1)(|X|-1)}[Y,[Z,X]]+(-1)^{(|Z|-1)(|Y|-1)}[Z,[X,Y]]=0 \tag{B.45}$$

which, using the graded antisymmetry above, can be written as the Loday identity:

$$[[X,Y],Z]=[X,[Y,Z]]-(-1)^{(|X|-1)(|Y|-1)}[Y,[X,Z]] \tag{B.46}$$

There is a graded Poisson Leibnitz rule

$$[X,Y\wedge Z]=[X,Y]\wedge Z+(-1)^{(|a|-1)|b|}Y\wedge[X,Z] \tag{B.47}$$

So this is a graded Poisson algebra with a -1 graded bracket, which is the definition of a Gerstenhaber algebra.

The following formula for the exterior derivative of a $|X|+|Z|-1$ form ω may also be used as a definition of the Schouten bracket on multivectors:

$$(-1)^{|X|}(\mathrm{d}\omega)(X\wedge Y)=Y(\omega(X))-(-1)^{(|X|-1)(|Z|-1)}X(\omega(Y))+\omega([X,Y]) \tag{B.48}$$

which may also be written

$$(-1)^{|X|}(X\wedge Y)\lrcorner\,\mathrm{d}\omega \;=\; Y\lrcorner\,\mathrm{d}(X\lrcorner\,\omega)-(-1)^{(|X|-1)(|Z|-1)}\;X\lrcorner\,\mathrm{d}(Y\lrcorner\,\omega)+[X,Y]\lrcorner\,\mathrm{d}\omega \tag{B.49}$$

This is a generalization of the formula for forms ω and vector fields X, Y, where the SN bracket is the Lie bracket of vector fields:

$$(-1)(\mathrm{d}\omega)(X\wedge Y)=Y(\omega(X))-X(\omega(Y))+\omega([X,Y]) \tag{B.50}$$

B.4.4 Frolicher-Nijenhuis bracket

The Frolicher-Nijenhuis bracket, $[x,y]^{FN}$, of vector valued forms $x,y\in\Omega^*(M,TM)$ is

$$[\omega\otimes X,\eta\otimes Y]^{FN} \;:=$$

$$\omega\wedge\eta\otimes[X,Y]+\omega\wedge\mathcal{L}_X\eta\otimes Y-\mathcal{L}_Y\omega\wedge\eta\otimes X+(-1)^{(|\omega|)}(d\omega\wedge i_X(\eta)\otimes Y+i_Y(\omega)\wedge d\eta\otimes X). \tag{B.51}$$

It can be defined via a subalgebra of the derivations of forms: $\mathcal{L}_{[x,y]^{FN}}=[\mathcal{L}_x,\mathcal{L}_y]$, which is the graded commutator of Lie derivations and the grade $|x|$ Lie derivation is defined by $\mathcal{L}_x:=[\mathrm{d},i_x]$, where i_x is a grade $|x|-1$ derivation. Vector valued forms form a graded Lie algebra with the Frolicher-Nijenhuis bracket, which extends the Lie bracket on vector fields.

B.4.5 Nijenhuis-Richardson bracket

$[x,y]^{NR}$ is the Nijenhuis-Richardson bracket of vector valued forms x and y, and is of grade degree $|x|+|y|$, where $x=\omega\otimes X, y=\eta\otimes Y$:

$$[x,y]^{NR} = i_x \cdot y - (-1)^{(|x|-1)(|y|-1)} i_y \cdot x \tag{B.52}$$

where $i_x \cdot y := \omega \wedge (X \lrcorner \eta) \otimes Y = i_x(\eta) \otimes Y$, where

$$i_x(\eta)(Z_{\sigma(1)}, Z_{\sigma(2)}, \ldots, Z_{\sigma(|\omega|+|\eta|+1)}) \ := $$
$$\frac{1}{(|x|)!(|\eta|-1)!} \sum_{\sigma\in S} (-1)^{|\sigma|} \eta(\omega(Z_{\sigma(1)}, Z_{\sigma(2)}, \ldots, Z_{\sigma(|\omega|)}) \otimes X, Z_{\sigma(|\omega|+1)}, \ldots, Z_{\sigma(|\omega|+|\eta|+1)}) \tag{B.53}$$

and $|y| = |\eta|$ is the degree of the form part η of y. This bracket has the following property in terms of the graded Lie algebra of derivations: $i_{[x,y]^{NR}} = [i_x, i_y]$. The commutator of an element of the Lie derivation subalgebra with an element of the generalized contraction derivation subalgebra is a mixed subalgebra type quantity $[\mathcal{L}_x, i_y] = i_{[x,y]^{FN}} - (-1)^{|x||y|} \mathcal{L}_{i_x(y)}$, so the sub Lie algebras are not ideals of the graded Lie algebra of derivations.

B.4.6 Batalin-Vilkovisky (BV) algebra

A BV algebra is also called an exact Gerstenhaber algebra.

If there is a degree -1 operator D acting on elements of a Gerstenhaber algebra such that

$$[A,B] = (-1)^{|A|}(D(A\wedge B) - (DA \wedge B + (-1)^{|A|} A \wedge DB)) \tag{B.54}$$

then D is said to generate the Gerstenhaber algebra, and the bracket measures the departure of D from the graded Leibnitz rule.

An exact Gerstenhaber algebra has the additional property $D^2 = 0$.

B.4.7 Buttin bracket

The Buttin bracket on multivector-valued forms which is equivalent to the ‘big bracket’, the canonical Poisson bracket on the ‘Dirac structure’ $\Lambda^\bullet(E^* \bigoplus E)$. On symplectic manifolds one has Poisson brackets on functions and Koszul brackets on differential forms: $[\omega,\xi]^\Pi := \omega \llcorner \Pi \lrcorner d\xi$. Note for symplectic form ω, $[\omega,\cdot]^{\omega^{-1}} = d\cdot$, the de Rham differential, and there is a derived Leibnitz (also called Loday) bracket: $[\![\alpha,\beta]\!] := [[\alpha,\omega]^{\omega^{-1}}, \beta]^{\omega^{-1}}$.

The Buttin bracket $[\xi \otimes v, \zeta \otimes w]_B$, where $\xi \otimes v, \zeta \otimes w \in \Lambda^\bullet E^* \bigotimes \Lambda^\bullet E$ is defined by mapping multivector-valued forms to $i_{\xi\otimes v} = \xi \wedge i_v$ in the space of differentials of forms in $\omega \in \Lambda^\bullet E^*$, where the multivector factor becomes an inner product and the form part becomes an exterior product generalizing the contraction with a vector valued form above. Then the Buttin bracket is the highest multivector-degree term from the graded commutator of differentials (which form a Lie algebra). A Loday algebra is a non graded antisymmetric generalization of a Lie algebra. A n-graded Loday algebra is an algebra with a (not necessarily graded antisymmetric) bilinear bracket which obeys the graded Jacobi identity in the form of the graded Leibnitz rule: $[a,[b,c]] = [[a,b],c] + (-1)^{(|a|+n)(|b|+n)}[b,[a,c]]$. Examples of Loday brackets are derived brackets: $[a,b]_D := (-1)^{(|a|+1+n)}[Da,b]$ where D is an odd differential, and $[,]$ a grade n graded Lie bracket.

$$i_{\xi\otimes v}(\omega) := \xi \wedge i_v(\omega) \tag{B.55}$$

$$i_{[\xi\otimes v,\zeta\otimes w]_B} := Max_{type}(\ [i_{\xi\otimes v}, i_{\zeta\otimes w}]\ ) \tag{B.56}$$

where Max_{type} gives the highest multivector-degree term of any element in $\Lambda^\bullet E^* \bigotimes \Lambda^\bullet E$

The Buttin bracket on multivector valued forms on a module E is a graded Lie bracket which extends the Nijenhuis-Richardson bracket.

B.4.8 Leibnitz (Loday) algebra

A generalization of a graded Lie algebra is a Leibnitz algebra (also called Loday algebra) of degree n, which does not generally have the graded anti-commutative property but does have the following form of the graded Jacobi identity:

$$[\![a,[\![b,c]\!]]\!] = [\![[\![a,b]\!],c]\!] + (-1)^{(n+|a|)(n+|b|)}[\![b,[\![a,c]\!]]\!] \tag{B.57}$$

An example is a construction for any graded Lie algebra $\mathfrak{U} \ni a, b$ with bracket of degree n and a differential D:

$$[\![a,b]\!]_{(D)} := (-1)^{n+|a|+1}[Da,b] \tag{B.58}$$

This is called the derived bracket of [] by D. The map D is a Leibnitz morphism from the this Leibnitz algebra to the original Lie algebra. It may be that D could be an interior derivation generated by d: $D\cdot = [d,\cdot]$ where d is an element of the Lie algebra with $[d,d] = 0$ and either d is odd or the Lie bracket is odd, in which case the derived graded Leibnitz bracket is $[\![a,b]\!]_d = [[a,d],b]$. An example is the Lie algebra of derivations of differential forms with the graded commutator. If we take d to be the deRham differential and a and b to be inner products with multivectors then the derived bracket above is the Schouten-Nijenhuis bracket on multivectors, which is a Gerstenhaber algebra.

A Poisson-Liebnitz algebra is a generalization of Poisson algebras but using a Liebnitz bracket: adjoint operators should be derivations on the associative algebra structure:

$$[\![a, bc]\!] = [\![a, b]\!]c + (-1)^{(n+|a|)|b|}\, b[\![a, c]\!] \tag{B.59}$$

$$[\![ab, c]\!] = a[\![b, c]\!] - (-1)^{(n+|a|)|b|}\, b[\![a, c]\!] \tag{B.60}$$

$$[\![ab, c]\!] = a[\![b, c]\!] + (-1)^{(n+|c|)|b|}[\![a, c]\!]b \tag{B.61}$$

B.4.9 Lie algebroid

Because one wants to consider various vector bundles over a base manifold, and have a Lie bracket on sections the natural structure to consider is the *Lie algebroid* A, which is a vector bundle over $\mathcal{M}$, with a Lie bracket $[,]_A : \Gamma A \bigoplus \Gamma A \longrightarrow \Gamma A$ where there is a vector-bundle map (called the *anchor*) $\rho : \mathsf{A} \longrightarrow \mathfrak{T}M$ and a Leibnitz rule for products of sections $X, Y \in \Gamma A$ and functions $f \in C^\infty(\mathcal{M}) : [X, fY]_A = f[X, Y]_A + (\rho(X) \cdot f)Y$. The anchor is in fact a Lie algebra homomorphism $\rho : \Gamma A, [,]_A \longrightarrow \Gamma \mathfrak{T}M, [,]$, the Lie bracket on vector-fields. When ρ is the identity we recover the case of the tangent bundle $A = \mathfrak{T}M, [,]$. When $\rho(A) = \{0\}$ we recover a collection of Lie algebras smoothly defined pointwise over $\mathcal{M}$. The Lie bracket can be extended to the algebraic Schouten bracket $[,]_s$ of multivectors $X, Y \in \Gamma(\Lambda^\bullet A) \cong C^\infty(\Pi A^*)$, which is an odd graded Poisson bracket of functions in $C^\infty(\Pi A^*)$. This is equal to the derived bracket of a quadratic Hamiltonian on $\Pi A^* : [X, Y]_A = \{\{X, H\}, Y\}$, where $H \in C^\infty(\mathfrak{T}^*(\Pi A^*)) \cong C^\infty(\Pi A^* \bigoplus \Pi A) \cong \Lambda^\bullet(A \bigoplus A^*)$ and $\{,\}$ is the canonical graded Poisson bracket, also known as the big bracket, and X, Y are pulled-back to and thus taken to be constant on the fibers of the cotangent space to ΠA^*, which is the bundle dual to A with the fiber coordinates given an odd grading. Explicitly

$$H(x^\mu, \bar{\alpha}_i, p_\mu, \bar{\theta}^i) = \rho_i^\mu(x) p_\mu \bar{\theta}^i + \frac{1}{2}\bar{\alpha}_k C_{ij}^k(x)\bar{\theta}^i\bar{\theta}^j \tag{B.62}$$

and $\{H, H\} = 0$. The Lie algebroid is also equivalent to a Poisson algebra structure of functions f, g on A^* where $\{f, g\}_{A^*} = [[f, P]_A, g]_A$, where P is the Poisson bivector field on A^* and f, g are pulled back to $\Gamma(\Lambda^\bullet A)$ and considered to be multivectors of degree 0. Explicitly,

$$P = \rho_i^\mu(x)\frac{\partial}{\partial x^\mu}\frac{\partial}{\partial \alpha^i} + \frac{1}{2}\alpha_k C_{ij}^k(x)\frac{\partial}{\partial \alpha^j}\frac{\partial}{\partial \alpha^i} \tag{B.63}$$

Considering the graded commutator $[,]$ of bundle endomorphisms of the exterior algebra of forms $\Gamma(\Lambda^\bullet A^*) \cong C^\infty(\Pi A)$, then: $i_{[X,Y]_A} = [[i_X, d_A], i_Y] = [\mathcal{L}_X, I_Y]$, and for $Y = f \in C^\infty(\mathcal{M})$ and $X \in \Gamma A$, $\rho(X) \cdot f = i_{[X,f]_A} = [[i_X, d_A], f]$, where $\mathcal{L}_X$ is the Lie derivative. The degree 1 differential d_A can be written explicitly in a coordinate basis as:

$$d_A = \bar{\alpha}^i \rho_i^\mu(x)\frac{\partial}{\partial x^\mu} + \frac{1}{2}\bar{\alpha}^j\bar{\alpha}^i C_{ij}^k(x)\frac{\partial}{\partial \bar{\alpha}^k}. \tag{B.64}$$

and $[d_A, d_A] = 0$. This differential on $C^\infty(\Pi A)$ can be considered as a vector field on ΠA, called a homological vector field or a Q-structure. Another bracket, the Frolicher-Nijenhuis bracket, can be defined on form valued vector fields on $\mathcal{M}$, $X, Y \in \Gamma(\Lambda^\bullet A^* \bigotimes A)$: $\mathcal{L}_{[X,Y]_{FN}} = [\mathcal{L}_X, \mathcal{L}_Y]$. The particular case when the algebroid is the tangent bundle is when $\rho_i^\mu = \delta_i^\mu$ and the case when it is a Lie algebra is when $\rho = 0$.

B.5 Schouten bracket and multisymplectic forms

In this section we will introduce the use of the Schouten bracket and multivectorfields in relation to hamiltonian forms and multisymplectic forms.

Multivector Lie derivatives of forms

We employ some definitions:

Definition: ‘almost-hamiltonian form’ $\theta_X := -(-1)^{|X|} X \lrcorner \Theta$.

Definition: ‘anti-hamiltonian form’ $\hat{\omega}_X := \hat{\omega}_X := X \lrcorner \mathrm{d}\Theta + \mathrm{d}\theta_X$.

Definition: ‘form complement to X’ $\omega_X := (-1)^{|X|} X \lrcorner \Omega$.

Definition: ‘Lie derivative of the form Θ relative to the multivector field X $\mathcal{L}_X\Theta := [i_X, \mathrm{d}]\Theta = i_X(\mathrm{d}\Theta) - (-1)^{|X|}\mathrm{d}(i_X\Theta)$.

Given a degree s alternating multivector field $X \in \mathfrak{X}^s(M)$ on an arbitrary manifold M, we define a form of degree $d-s$, the ‘almost-hamiltonian form to the multivector X relative to the form Θ’, $\theta_X := -(-1)^{|X|} X \lrcorner \Theta$, where Θ is any form of degree d (not related to the dimension of M) on M.

Then the generalized Lie derivative of Θ relative to the multivector X is:

$$\mathcal{L}_X\Theta := [i_X, \mathrm{d}]\Theta = i_X(\mathrm{d}\Theta) - (-1)^{|X|}\mathrm{d}(i_X\Theta) = -X \lrcorner \Omega + \mathrm{d}\theta_X =: \hat{\omega}_X \tag{B.65}$$

where Ω is defined by $\Omega = -\mathrm{d}\Theta$. We also have the ‘anti-hamiltonian form to the multivector relative to the form Θ, $\hat{\omega}_X := -X \lrcorner \Omega + \mathrm{d}\theta_X$

If $\Omega = -\mathrm{d}\Theta$ were an exact multisymplectic form, $\mathcal{L}_X\Theta$ would be the obstacle for the ‘theta complement to X’, $\theta_X := -(-1)^{|X|} X \lrcorner \Theta$, to be the hamiltonian $(d-s)$-form corresponding to the s-multivector field X. In particular, if X is a vector field ($s = 1$), we call X an *exact* *multisymplectomorphism*, with corresponding hamiltonian $d-1$-form θ_X, if $\mathcal{L}_X\Theta = 0$. This is a multisymplectomorphism because $\mathcal{L}_X\Theta = 0$ implies $0 = \mathrm{d}\mathcal{L}_X\Theta = \mathcal{L}_X\mathrm{d}\Theta = \mathcal{L}_X\Omega$.

The generalized Lie derivative of a closed form Ω relative to the multivector X is:

$$\mathcal{L}_X\Omega := [i_X, \mathrm{d}]\Omega = X \lrcorner \mathrm{d}\Omega - (-1)^{|X|}\mathrm{d}(X \lrcorner \Omega) = -(-1)^{|X|}\mathrm{d}(X \lrcorner \Omega) =: -\mathrm{d}\omega_X \tag{B.66}$$

where the 'form Ω-complement to X ', ω_X, is shorthand for $(-1)^{|X|}X \lrcorner \Omega$.

If Ω is exact, $\Omega = -\mathrm{d}\Theta$,

$$\mathcal{L}_X\Omega = -\mathcal{L}_X\mathrm{d}\Theta = -\mathrm{d}\mathcal{L}_X\Theta = -\mathrm{d}(X\lrcorner\mathrm{d}\Theta - (-1)^{|X|}\mathrm{d}(X\lrcorner\Theta)) = \mathrm{d}(X\lrcorner\Omega - \mathrm{d}\theta_X) = \mathrm{d}(X\lrcorner\Omega = -\mathrm{d}\hat{\omega}_X \tag{B.67}$$

$\mathcal{L}_X\Omega$ is the obstruction for $\omega_X := (-1)^{|X|}X \lrcorner \Omega$ to be closed. In the latter case, if Ω were a multisymplectic form, X would be a locally hamiltonian multivector field and V would be the obstruction to X being a locally hamiltonian multivector field. If $\omega_X := -(-1)^{|X|}X \lrcorner \Omega$ was also exact, then X would be a hamiltonian s-multivector field and h_X, where $\mathrm{d}h_X = \omega_X$, would be the corresponding hamiltonian $d-s$-form. If X is Ω-locally-hamiltonian, then $\mathcal{L}_X\Omega = 0$.

Schouten bracket

Defined for multivector fields $X, Y \in \mathfrak{X}^\bullet(M)$, is the *Schouten bracket* $[X, Y]_{sh}$, described earlier in this appendix, which is a Gerstenhaber algebra structure (odd graded Lie algebra) on (alternating) multivector fields, which is a natural generalization on the Lie bracket of vector fields. We also have $[X, Y]_{sh} = \mathcal{L}_X Y$ which is the generalization of the Lie derivative to multivector fields.

The following is the Schouten bracket identity (B.48) (which could also serve as a definition of a Schouten bracket) for any d-form Θ, and any multivector fields X , Y on a manifold where $\deg(\Theta) = |X| + |Y| - 1 = d$ (d is not related to the dimension of M):

$$(-1)^{|X|}(X \wedge Y) \lrcorner \mathrm{d}\Theta \;=\; Y \lrcorner \mathrm{d}(X \lrcorner \Theta) - (-1)^{(|X|-1)(|Y|-1)}\, X \lrcorner \mathrm{d}(Y \lrcorner \Theta) + [X, Y]_{sh} \lrcorner \Theta \tag{B.68}$$

where the result on both sides is a 0-form (i.e. function on the manifold M).

We employ the above with $\Omega = -\mathrm{d}\Theta$,

$$\begin{aligned} &-(-1)^{|X|}(X \wedge Y) \lrcorner \Omega \\ &= \; -(-1)^{|X|}Y \lrcorner \mathrm{d}\theta_X + (-1)^{|Y|}(-1)^{(|X|-1)(|Y|-1)}\, X \lrcorner \mathrm{d}\theta_Y + [X, Y]_{sh} \lrcorner \Theta \\ &= \; -(-1)^{|X|}Y \lrcorner \mathrm{d}\theta_X + (-1)^{|Y|}(-1)^{(|X|-1)(|Y|-1)}\, X \lrcorner \mathrm{d}\theta_Y - (-1)^{|X|+|Y|-1}\theta_{[X,Y]_{sh}} \end{aligned} \tag{B.69}$$

$$(X \wedge Y) \lrcorner \Omega \ =$$
$$Y \lrcorner \mathrm{d}\theta_X + (-1)^{(|X||Y|)}\ X \lrcorner \mathrm{d}\theta_Y - (-1)^{|Y|}\theta_{[X,Y]_{sh}} \tag{B.70}$$

For a multisymplectic form Ω, if X and Y are hamiltonian $(X \wedge Y) \lrcorner \Omega \ = \{h_X, h_Y\}$, and the above can be written

$$\{h_X, h_Y\} \ = \ Y \lrcorner \mathrm{d}\theta_X - (-1)^{(|X|-1)(|Y|-1)}(-1)^{|X|}(-1)^{|Y|}\ X \lrcorner \mathrm{d}\theta_Y - (-1)^{|X|}[X,Y]_{sh} \lrcorner \Theta \tag{B.71}$$

If X and Y are exact hamiltonian, then this can be written

$$\{h_X, h_Y\} \ = \ \{h_Y, h_X\} + (-1)^{|X||Y|}\{h_X, h_Y\} - (-1)^{|X|}[X,Y]_{sh} \lrcorner \Theta \tag{B.72}$$

using $\{h_Y, h_X\} = (-1)^{|X||Y|}\{h_X, h_Y\}$ we obtain

$$\{h_X, h_Y\} \ = -(-1)^{|X|+|Y|-1}[X,Y]_{sh} \lrcorner \Theta \ = -(-1)^{d}[X,Y]_{sh} \lrcorner \Theta = h_{[X,Y]_{sh}} \tag{B.73}$$

In the particular case of an exact hamiltonian vector field X, $s = 1$, and a hamiltonian d-multivector field $Y \in \mathfrak{X}^d(\mathcal{M})$, with corresponding $d-1$ and 0 -forms $T = h_X$ and $\mathcal{H} = h_Y$ respectively, this gives:

$$\{h_X, h_Y\} \ = \{T, \mathcal{H}\} \ = -(-1)^{|X|+|Y|-1}[X,Y]_{sh} \lrcorner \Theta \ = -(-1)^{|X|+|Y|-1}(\mathcal{L}_X Y) \lrcorner \Theta$$
$$= (\mathcal{L}_Y X) \lrcorner \Theta = \theta_{\mathcal{L}_X Y} = \theta_{\delta_X Y} = h_{[X,Y]_{sh}} = \theta_{\mathcal{L}_X Y} = \theta_{[X,Y]_{sh}} = X \lrcorner \mathrm{d}\mathcal{H} = \mathcal{L}_X \mathcal{H} = \delta_X \mathcal{H} \tag{B.74}$$

the infinitesimal variation in the function $\mathcal{H}$ from the flow of the vector field X.

If $\mathcal{L}_X \Theta = 0 = \mathcal{L}_Y \Theta$ this can be also written as

$$-(-1)^{|X|}(X \wedge Y) \lrcorner \Omega \ =$$
$$-(-1)^{|X|}Y \lrcorner X \lrcorner \Omega \ + (-1)^{(|X|-1)(|Y|-1)}\ (-1)^{|Y|}X \lrcorner Y \lrcorner \Omega + [X,Y]_{sh} \lrcorner \Theta \tag{B.75}$$

So

$$(X \wedge Y) \lrcorner \Omega \ = \ Y \lrcorner X \lrcorner \Omega \ - (-1)^{(|X|-1)(|Y|-1)}\ (-1)^{|X|}(-1)^{|Y|}X \lrcorner Y \lrcorner \Omega - (-1)^{|X|}[X,Y]_{sh} \lrcorner \Theta \tag{B.76}$$

$$(X \wedge Y) \lrcorner \Omega \ = \ Y \lrcorner X \lrcorner \Omega \ + (-1)^{|X||Y|}X \lrcorner Y \lrcorner \Omega - (-1)^{|X|}[X,Y]_{sh} \lrcorner \Theta = -(-1)^{|X|}[X,Y]_{sh} \lrcorner \Theta \tag{B.77}$$

Employing the Schouten bracket identity (B.48) on the form $\Omega = -\mathrm{d}\Theta$ and any multivector fields X , Y where where $\deg(\Omega) = |X| + |Y| - 1$.

$$(-1)^{|X|}(X \wedge Y) \lrcorner \mathrm{d}\Omega \ = \ Y \lrcorner \mathrm{d}(X \lrcorner \Omega) - (-1)^{(|X|-1)(|Y|-1)}\ X \lrcorner \mathrm{d}(Y \lrcorner \Omega) + [X,Y]_{sh} \lrcorner \Omega \tag{B.78}$$

$$0 = \; Y \lrcorner \, \mathrm{d}(X \lrcorner \, \Omega) - (-1)^{(|X|-1)(|Y|-1)} \; X \lrcorner \, \mathrm{d}(Y \lrcorner \, \Omega) + [X,Y]_{sh} \lrcorner \, \Omega \tag{B.79}$$

$$0 = \; -(-1)^{(|X|)} Y \lrcorner \, \mathrm{d}\omega_X + (-1)^{(|Y|)}(-1)^{(|X|-1)(|Y|-1)} \; X \lrcorner \, \mathrm{d}\omega_Y + [X,Y]_{sh} \lrcorner \, \Omega \tag{B.80}$$

$$(-1)^{(|X|+|Y|-1)} \omega_{[X,Y]_{sh}} = \; -(-1)^{(|X|)} Y \lrcorner \, \mathrm{d}\omega_X + (-1)^{(|Y|)}(-1)^{(|X|-1)(|Y|-1)} \; X \lrcorner \, \mathrm{d}\omega_Y \tag{B.81}$$

$$(-1)^{(|Y|)} \omega_{[X,Y]_{sh}} = \; Y \lrcorner \, \mathrm{d}\omega_X + (-1)^{|X||Y|} \; X \lrcorner \, \mathrm{d}\omega_Y \tag{B.82}$$

This can be written as

$$\begin{aligned}
&(-1)^{(|Y|)} \omega_{\mathcal{L}_X Y} \\
&= \; \mathcal{L}_Y \omega_X + (-1)^{(|X||Y|)} \mathcal{L}_X \omega_Y + (-1)^{|Y|} \; \mathrm{d}(Y \lrcorner \, \omega_X) + (-1)^{|X|}(-1)^{|X||Y|} \mathrm{d}(X \lrcorner \, \omega_Y) \\
&= \; \mathcal{L}_Y \omega_X + (-1)^{(|X||Y|)} \mathcal{L}_X \omega_Y - (-1)^{|Y|}(-1)^{|X|} \; \mathrm{d}(Y \lrcorner \, X \lrcorner \, \Omega) \\
&\quad -(-1)^{|Y|}(-1)^{|X|}(-1)^{|X||Y|} \mathrm{d}(X \lrcorner \, Y \lrcorner \, \Omega) \\
&= \; \mathcal{L}_Y \omega_X + (-1)^{(|X||Y|)} \mathcal{L}_X \omega_Y - (-1)^{|Y|}(-1)^{|X|} \; \mathrm{d}(Y \lrcorner X \lrcorner \Omega) - (-1)^{|Y|}(-1)^{|X|} \mathrm{d}(Y \lrcorner X \lrcorner \Omega)
\end{aligned} \tag{B.83}$$

The Schouten brackets obey a super Jacobi identity and form a super Poisson structure on multivector fields, as explained at the beginning of this chapter, which may be of use in extending the Poisson bracket to multisymplectic manifolds.

Appendix C

The electromagnetic field in phase space

For the purposes of describing the structure of the Lagrangian and Hamiltonian formalisms for classical mechanics, in particular to show how gauge symmetries are handled, we present the example of the electromagnetic field (without interactions) in Minkowski space. The use of a field as an example was chosen to allow comparison with the multiphase-space formalism which is designed specifically for local field theories. Nevertheless most of the features emphasized in this section could have been described by slightly simpler examples, such as a constrained particle. Basic information from [12].

The electromagnetic field in 4-dimensional Minkowski M^4 space is initially defined by a configuration space Lagrangian, where the Lagrangian is a function of a 1-form field $A = A_\mu \mathrm{d}x^\mu$ in a time slice and the first time derivative of the field, so that there are explicitly 4 degrees of freedom (A_t, A_x, A_y, A_z) for each spacetime point, corresponding to a phase space which has 8 dimensions at each spatial point.

As will be shown below, because of the particular form of this Lagrangian, some of these degrees of freedom are non-physical and the equations of motion either constrains them to values determined by the physical degrees of freedom, or do not determine them at all, so that they can be varied without changing the physical observables, which are determined. Nevertheless the non-physical variables are of use, for example in retaining in the formalism the manifest covariance of the 1-form A.

C.1 Lagrangian with gauge symmetry

We start with a Lagrangian of the electromagnetic field in Minkowski space in Cartesian coordinates with metric $\eta^{\mu\nu}$ .The Lagrangian is a function of fields which are the electromagnetic potentials $(A_\mu) = (A_t, \vec{A}) = (A_t, A_x, A_y, A_z)$, which can be viewed as the components of a 1-form field $A(x) = A_\mu(x)\mathrm{d}x^\mu$ on Minkowski spacetime.

$$
\begin{aligned}
L(A_\mu(x), \partial_\nu A_\mu(x), x) &= \int |\mathrm{d}A|^2 \mathrm{d}^3x = \int \partial_{[\mu}A_{\nu]}\ \partial_{[\lambda}A_{\rho]}\ g^{\mu\lambda}g^{\nu\rho}\ \mathrm{d}^3x \\
= \frac{1}{2}\int [(\partial_t\vec{A} - \vec{\nabla}A_t)^2 - \vec{B}^2]\ \mathrm{d}^3x &= \frac{1}{2}\int\ [\vec{E}\cdot(\partial_t\vec{A} - \vec{\nabla}A_t) - \vec{B}^2]\ \mathrm{d}^3x \\
&= \frac{1}{2}\int\ [\vec{E}\cdot\partial_t\vec{A} + A_t\vec{\nabla}\cdot\vec{E} - \vec{B}^2]\ \mathrm{d}^3x
\end{aligned} \tag{C.1}
$$

where we use the shorthand $\vec{B} := \vec{\nabla}\times\vec{A}$ and $\vec{E} := \partial_t\vec{A} - \vec{\nabla}A_t$. For the last equality we used integration by parts and are ignoring the resulting surface term.

The Lagrangian density has a local symmetry, i.e it is invariant under a one parameter variation which can be arbitrary at each point in spacetime.

The infinitesimal gauge variation is $\delta A_t = \partial_t f(x), \delta A_i = \partial_i f(x)$ which is $\delta A = \mathrm{d}f$ in the notation of forms, with $f(x)$ is the variation parameter which is an arbitrary smooth function on spacetime. Both $\vec{E}$ and $\vec{B}$ are invariant under this transformation, and therefore so is the Lagrangian density $\frac{1}{2}(\vec{E}^2 - \vec{B}^2)$. This results in that, of the original 4 degrees of freedom (A_t, A_x, A_y, A_z), there are no more than 3 *physical* degrees of freedom, with the extra, arbitrarily variable, gauge degree of freedom being non physical. This is because the action principle $\delta S = 0$ does not fix the extra degree of freedom, simply as a result of the invariance of the Lagrangian density under arbitrary variation in this gauge degree of freedom.

In the Lagrangian (C.1) above there is no functional dependence on $\partial_t A_t$, so the Euler-Lagrange equation for A_t has the form of a constraint $\vec{\nabla}\cdot\vec{E} \approx 0$ with the time component A_t of the 4-potential A in the role of a Lagrange multiplier, as can be seen from (C.1).

The presence of constraints is a general consequence of gauge degrees of freedom: If λ_a are the gauge degrees of freedom and we change the coordinate system of the degrees of freedom so that λ_a are a subset of them and the others are physical coordinates invariant under gauge variations (which only affect λ_a), then the gauge invariance conditions, $\frac{\delta L}{\delta\lambda_a} = 0$, are also the Euler-Lagrange equations for the gauge degrees of freedom, and are constraint type equations, rather than differential equations in time.

Thus the gauge symmetry leads to constraints (in this example, Gauss' law) on the solutions of the variational principle. These also apply to initial conditions.

C.1.1 Legendre transformation

The phase space here has a set of coordinates $(A_t, A_x, A_y, A_z, P^t, P^x, P^y, P^z) = (A_t, \vec{A}, P^t, \vec{P})$ at each point (t, y, z) in space. The Legendre transformation of the Lagrangian (C.1) is

$$\vec{P} \approx \vec{E} \quad , \quad P^t \approx 0 \quad , \tag{C.2}$$

$$H \;=\; \int [\, \frac{1}{2}\vec{P}^2 + \frac{1}{2}\vec{B}^2 - A_t\vec{\nabla}\cdot\vec{P} + P^t\partial_t A_t \,]\, \mathrm{d}^3x \;=\; \int [\, \frac{1}{2}\vec{P}^2 + \frac{1}{2}\vec{B}^2 - A_t(\vec{\nabla}\cdot\vec{P} + \partial_t P^t)\,]\, \mathrm{d}^3x \tag{C.3}$$

is the Hamiltonian $H = P^\mu \partial_t A_\mu - L$. where integration by parts is used for the last term in the integrand, and the boundary term is ignored. There is a primary constraint, $P^t \approx 0$ above, which results in that $\partial_t A_t$ cannot be expressed in terms of P^t - so that the last term in the integrand ends up containing a time derivative of a field, $\partial_t A_t$. This is a general feature of non-invertible Legendre transformations. Ultimately this is because the time derivative $\partial_t A_t$ is not present in the configuration Lagrangian (C.1). This leads to the Hamilton's equation of motion for the A_t and P^t to fail to give the time derivatives.

C.1.2 Hamiltonian system with constraint

We now take our starting point a dynamical system defined by the electromagnetic field Hamiltonian H above together with the constraint $P^t \approx 0$. For a solution of Hamilton's equations to be consistent with the primary constraint we need the constraint to hold along the time evolution of a solution: $0 = \dot{P}^t \approx \{P_t, H\} = \vec{\nabla}\cdot\vec{P} + \partial_t P^t = \partial_\mu P^\mu =: \mathcal{P}$. Thus we have a secondary constraint $\vec{\nabla}\cdot\vec{P} \approx 0$, where we have set $\partial_t P^t$ equal to zero employing the primary constraint. We need to check that the secondary constraint is compatible with Hamilton's equation: $\frac{\mathrm{d}}{\mathrm{d}t}(\vec{\nabla}\cdot\vec{P}) \approx \{\vec{\nabla}\cdot\vec{P}, H\} = -\vec{\nabla}\cdot\{\vec{P}, H\} = \vec{\nabla}\cdot(\vec{\nabla}\times\vec{B}) = 0$. Thus the secondary constraint is preserved over time. So there are no further constraints needed in this model.

To determine the time evolution instantaneous rate of change in phase space we employ Hamilton's equations in the form of the Poisson bracket with the Hamiltonian:

(1) The Hamilton's equation of motion for A_t is undetermined: $\dot{A}_t \approx \{A_t, H\} = \partial_t A_t$.

(2) The Hamilton's equation of motion for P^t is : $\dot{P}^t \approx \{P^t, H\} \;= \vec{\nabla}\cdot\vec{P} + \partial_t P^t = \mathcal{P}$. On the constraint surface $\mathcal{P} = 0$ the time derivative of P^t is zero, which is just the consistency condition for the primary constraint.

(3) The Hamilton's equation of motion for A_i is : $\dot{\vec{A}} \approx \{\vec{A}, H\} \;=\; \vec{P} \;+\; \vec{\nabla}A_t$.

(4) The Hamilton's equation of motion for P^i is : $\dot{\vec{P}} \approx \{\vec{P}, H\} \;=\; \vec{\nabla}\times\vec{B}$.

The Hamilton's equation of motion show that the dynamical system is determined by two equations: equation (4), $\dot{\vec{P}} \approx \{\vec{P}, H\} \;=\; \vec{\nabla} \times \vec{B}$, and the secondary constraint, $\vec{\nabla} \cdot \vec{P} \approx 0$.

Equation (1) shows that A_t is undetermined and can be an arbitrary function, and equation (3) requires A_t to be fixed to give a relation between the time derivatives of the configuration variables A_i and their canonical momenta. The phase-space configuration observable A_t and its canonical conjugate P^t can be viewed as non-physical, and can be made constant in time, leaving the A_i and their canonical momenta P^i as the physical phase-space coordinates, whose instantaneous time evolution are determined by the Hamilton's equations, whereas the non-physical phase-space coordinates are either fixed (P^t) or not determined (A_t).

It is of interest to examine the constraints and the gauge variation and the link between them, in particular the role of Poisson brackets, the meaning of gauge variation in the phase-space setting and the consistency of the variation with time evolution.

C.1.3 Role of Poisson brackets

We first calculate the Poisson brackets of the constraints with the phase-space coordinates:

$$\{f(x)(\vec{\nabla} \cdot \vec{P} + \partial_t P^t), A_i\} = -\vec{\nabla} f(x) \cdot \{\vec{P}, A_i\} = -\nabla_i f(x)\{P^i, A_i\} = \nabla_i f(x) \tag{C.4}$$

(no sum over i index)

$$\{f(x)(\vec{\nabla} \cdot \vec{P} + \partial_t P^t), A_t\} = -\partial_t f(x)\{P^t, A_t\} = \partial_t f(x) \tag{C.5}$$

$$\{f(x)(\vec{\nabla} \cdot \vec{P} + \partial_t P^t), P^\mu\} = 0 \tag{C.6}$$

Summarizing, $\delta_f A_\mu = \{f(x)\mathcal{P}, A_\mu\} = \partial_\mu f(x)$, so the constraints generate the gauge variation which leaves the Lagrangian density invariant. We examine what the gauge variation signifies in the Hamiltonian setting. Because the generator is a function here of just the momenta, the momenta are invariant under the gauge variation, because the Poisson bracket of two momenta is zero. The gauge variation of the Hamiltonian density is $\delta_f \mathcal{H} = \{f(x)\mathcal{P}, \mathcal{H}\} = -(\partial_t f(x))\mathcal{P}$, which is zero in the case of global variation $\partial_t f(x) = 0$, or on the constraint surface $\mathcal{P} = 0$ in time extended phase space.

We examine the Poisson brackets of the constraints with the Hamiltons' equations:

$$\delta \dot{A}_t = \partial_t \dot{f}(x) \approx \{f(x)\mathcal{P}, \{A_t, H\}\} = \{f(x)\mathcal{P}, \partial_t A_t\} = \partial_t \partial_t f(x) \tag{C.7}$$

$$\delta \dot{A}_i = \partial_i \dot{f}(x) \approx \{f(x)\mathcal{P}, \{A_i, H\}\} = \{f(x)\mathcal{P}, P^i \;+\; \partial_i A_t\} = \partial_i \partial_t f(x) \tag{C.8}$$

$$\delta \dot{P}^\mu \approx \{f(x)\mathcal{P}, \{P^\mu, H\}\} = \{\mathcal{P}, -(\partial_t f(x))\mathcal{P}\} = 0 \tag{C.9}$$

So the variation is compatible with Hamilton's equations. These can also be calculated via the Jacobi identity:

$$\delta \dot{A}_\mu \approx \{f(x)\mathcal{P}, \{A_\mu, H\}\} = \{\{f(x)\mathcal{P}, A_\mu\}, H\} \ + \ \{A_\mu, \{f(x)\mathcal{P}, H\}\} =$$

$$= \{\partial_\mu f(x), H\} \ + \ \{A_\mu, -(\partial_t f(x))\mathcal{P}\} = 0 + -(-\partial_\mu \partial_t f(x)) = \partial_\mu \partial_t f(x) \tag{C.10}$$

$$\delta \dot{P}^\mu \approx \{f(x)\mathcal{P}, \{P^\mu, H\}\}$$

$$= \{\{f(x)\mathcal{P}, P^\mu\}, H\} \ + \ \{P^\mu, \{f(x)\mathcal{P}, H\}\} = 0 \ + \ \{P^\mu, -(\partial_t f(x))\mathcal{P}\}$$

$$= -(\partial_t f(x))\{P^\mu, \mathcal{P}\} = 0 \tag{C.11}$$

C.1.4 Phase-space Lagrangian system with constraint

The phase-space Lagrangian constructed from the Hamiltonian above, is

$$L_P(A_\mu(x), \partial_\nu A_\mu(x), P^\mu(x)) = \int [\ P^\mu \partial_t A_\mu - \mathcal{H}\ ]\ \mathrm{d}^3x$$

$$= \int [\ P^\mu \partial_t A_\mu \ - \ (\ \frac{1}{2}\vec{P}^2 \ + \ \frac{1}{2}\vec{B}^2 \ - A_t \vec{\nabla}\cdot\vec{P} \ + P^t \partial_t A_t\ )\ ]\ \mathrm{d}^3x$$

$$= \int [\ P^i \partial_t A_i \ - \ (\ \frac{1}{2}\vec{P}^2 \ + \ \frac{1}{2}\vec{B}^2 \ - A_t \vec{\nabla}\cdot\vec{P}\ )\ ]\ \mathrm{d}^3x$$

$$= \int [\ P^i(\partial_t A_i - \partial_i A_t) \ - \ (\ \frac{1}{2}\vec{P}^2 \ + \ \frac{1}{2}\vec{B}^2\ )\ ]\ \mathrm{d}^3x \tag{C.12}$$

where integration by parts is used to move the ∇: $A_t \vec{\nabla}\cdot\vec{P} = -P^i \partial_i A_t$ up to a boundary term.

This Lagrangian density is invariant under the local infinitesimal variation $\delta_f A_\mu = \partial_\mu f(x), \delta_f P^i = 0$ because $\delta_f(\partial_t A_i - \partial_i A_t) = (\partial_t \delta_f A_i - \partial_i \delta_f A_t) = (\partial_t \partial_i f - \partial_i \partial_t f) = 0$. In comparison, the gauge variation of the Hamiltonian density is $\delta_f \mathcal{H} = \{f(x)\mathcal{P}, \mathcal{H}\} = -(\partial_t f(x))\mathcal{P}$ - the extra term in L_P restores the gauge invariance. When the Euler-Lagrange equations, $\partial_t A_i \ - \ \eta_{ij} P^j \approx 0$ for the momenta P^i are substituted for P^i in the phase-space Lagrangian, the result is the configuration space Lagrangian (C.1). So the gauge invariance is extended from the configuration space Lagrangian density to the phase-space Lagrangian density.

The Euler-Lagrange equations for L_P are the Hamilton's equations (3) and (4) for $\vec{A}$ and $\vec{P}$, and the secondary constraint $\vec{\nabla}\cdot\vec{P} = 0$.

Because of the fact that the Euler-Lagrange equations for L_P are the physical Hamilton's equations, the discussion of the Euler-Lagrange equations is very much the same as the one for the Hamilton's equations:

There is no Euler-Lagrange equation for P^t because P^t is not present in L_P. The conjugate A_t is present as a non dynamical Lagrange multiplier whose Euler-Lagrange equation is the primary constraint $\vec{\nabla} \cdot \vec{P} = 0$ for this Lagrangian. This was a secondary constraint for the configuration space Lagrangian, whose primary constraint was $P^t = 0$. The only role that $P^t = 0$ had in the Hamiltonian scheme above was to produce a secondary constraint $\vec{\nabla} \cdot \vec{P} = 0$ via the requirement of $\dot{P}^t \approx \{P^t, H\} \ = 0$ along a trajectory. Otherwise P^t does not appear in the Hamilton's equations (3) and (4). In the Euler-Lagrange equations for L_P, P^t is not present and is indeed not needed because the physical constraint $\vec{\nabla} \cdot \vec{P} = 0$ does appear as one of the Euler-Lagrange equations.

Appendix D

Other multiphase-space examples

D.1 Scalar fields with global symmetry

The action of scalar fields $\bar{q}(x) = (q^1(x), q^2(x), \ldots, q^N(x))$ with $\bar{q}^2 := q^i q^j \delta_{ij} = (q^1)^2 + (q^2)^2 + \ldots + (q^N)^2$ is

$$S[q^i(x)] = \int_{M^d} \mathcal{L}(q^i, \partial_\mu q^i, x) \;\; \mathrm{d}^d x = \int_{M^d} \frac{m}{2} \, |\mathrm{d}\bar{q}|^2 - V(\bar{q}^2, x) \; \mathrm{d}^d x$$
$$= \int_{M^d} \frac{m}{2} \, \partial_\mu q^i \;\, \partial_\nu q^j \;\, g^{\mu\nu}(x) \delta_{ij} - V(\bar{q}^2) \; \mathrm{d}^d x \tag{D.1}$$

where the potential term $V(\bar{q}^2, x)$ is a function of $\bar{q}^2$ and spacetime position x.

The multiphase-space Legendre transformation is:

$$p_i^\mu(x) \approx m \, \partial_\nu q^j(x) g^{\nu\mu} \delta_{ij} \quad \text{and } \mathcal{H} \;=\; \frac{1}{2m} \bar{p}^2 \;+\; V(\bar{q}^2) \;=\; \frac{1}{2m} p_i^\mu p_j^\nu \; g_{\mu\nu} \delta^{ij} \;+\; V(\bar{q}^2) \tag{D.2}$$

$p_i^0(x)$ is the ordinary canonical momentum of the field q^i at the spacetime point x and $p_i^\kappa(x)$ is the stress, in the κ coordinate line direction, of the field q^i at the point (x).

The following is multiphase-space action,

$$S_{MP}[q^i(x), p_i^\mu(x)] = \int_{M^d} \mathcal{L}_{MP}(q^i, \partial_\mu q^i, p_i^\mu) \; \mathrm{d}^d x = \int_{M^d} \partial_\mu q^i \;\, p_i^\mu - \mathcal{H} \;\; \mathrm{d}^d x =$$
$$= \int_{M^d} \partial_\mu q^i \;\, p_i^\mu - \;\, (\frac{1}{2m} g_{\mu\nu} p_i^\mu p_j^\nu \delta^{ij} + V(\bar{q}^2)) \; \mathrm{d}^d x \tag{D.3}$$

The Euler-Lagrange equations for this are the DDW equations for the Legendre transformation of (D.1).

D.1.1 DDW Hamiltonian

The DDW equations of motion for the free scalar field using the the DDW Hamiltonian above, $\mathcal{H} = \frac{1}{2m} p_i^\mu p_j^\nu \; g_{\mu\nu}\delta^{ij} \; + V(\bar{q}^2)$, expressed using multi-Poisson brackets are, as expected,

$$0 \;\approx\; E_{q^i} := -\partial_\nu p_i^\nu - \frac{1}{d}\{p_i^\nu, \mathcal{H}\}_\nu = -\partial_\nu p_i^\nu - \frac{\partial \mathcal{H}}{\partial q^i} = -\partial_\nu p_i^\nu - 2V'(\bar{q}^2)q^i \text{ and} \tag{D.4}$$

$$0 \;\approx\; E_{p_i^\mu} := \partial_\mu q^i - \{q^i, \mathcal{H}\}_\mu = \partial_\mu q^i - \frac{\partial \mathcal{H}}{\partial p_i^\mu} = \partial_\mu q^i - \frac{1}{m}\; g_{\mu\nu}\delta^{ij} p_j^\nu \tag{D.5}$$

D.1.2 Symmetries

There is clearly a global symmetry, where the Lagrangian density is unchanged by any rotation $R \in SO(N)$, $\bar{q}' = R \cdot \bar{q}$ in the target space $\bar{q}(x) = (q^1(x), q^2(x), \ldots, q^N(x))$ because $\bar{q}^2 := q^i q^j \delta_{ij} = (q^1)^2 + (q^2)^2 + \ldots + (q^N)^2$ is the invariant for $SO(N)$ and both terms in the Lagrangian density are invariant, if the rotation R commutes with the spacetime partial derivatives. We consider in this subsection that this condition holds, i.e. R is constant on spacetime, and therefore called a global symmetry variation. If the symmetry were to hold for R arbitrarily varying (smoothly) with position, then the symmetry is called a local or gauge symmetry with gauge group $SO(N)$. With the local symmetry the symmetry group would be $M^d \times SO(N)$ and this is considered below in the sections dealing with the electromagnetic and Yang-Mills fields.

We consider infinitesimal variations $Y \in so(N)$, the Lie algebra for $SO(N)$. A basis for the $N(N-1)/2$ dimensional lie algebra $so(N)$ are the anti-symmetric matrices M_{rs} with matrix elements $(M_{rs})_{ij} = -\delta_{ri}\delta_{sj} + \delta_{rj}\delta_{si}$ where the infinitesimal variation $\bar{q}' = \bar{q} + \delta_Y \bar{q}$ of the fields $\bar{q}$ is given by $Y^{rs}\delta_{rs}q^i = Y^{rs}(M_{rs})_{ij}q^j$. The commutation relations of this basis of the Lie algebra $so(N)$ is

$$[M_{ij}, M_{km}] \;=\; \delta_{ik}M_{jm} + \; \delta_{jm}M_{ik} - \; \delta_{im}M_{jk} - \; \delta_{jk}M_{im} \tag{D.6}$$

For example $\delta_{12}q^1 = (M_{12})_{1j}q^j = 1q^2 = q^2$ and $\delta_{12}q^2 = (M_{12})_{2j}q^j = -1q^1 = -q^1$ because the only non-zero matrix elements of (M_{rs}) are $(M_{rs})_{rs} = 1$ and $(M_{rs})_{sr} = -1$, the infinitesimal rotation in the plane rs. We extend the variation to multimomenta on multiphase space

$$\delta_{rs}p_i^\mu(x) = (M_{rs})_{ji}p_j^\mu(x) = (M_{rs}^T)_{ij}p_j^\mu(x) = -(M_{rs})_{ij}p_j^\mu(x) \tag{D.7}$$

by using the Legendre transformation above, $p_i^\mu(x) \approx m\,\partial_\nu q^j(x) g^{\nu\mu}\delta_{ij}$, we see this is consistent on-shell with the variation of q:

$$\begin{aligned}\delta_{rs}p_i^\mu(x) &\approx m\,\partial_\nu(\delta_{rs}q^j(x))g^{\nu\mu}\delta_{ij} = m\,\partial_\nu((M_{rs})_{jk}q^k(x))g^{\nu\mu}\delta_{ij} \\ &= (M_{rs})_{ji} m\,\partial_\nu q^k(x) g^{\nu\mu}\delta_{jk} \approx (M_{rs})_{ji}p_j^\mu(x)\end{aligned} \tag{D.8}$$

because ∂_ν and δ_{jk} commute.

The variation of the covariant multiphase-space action due the infinitesimal variation, $\delta p_i^\mu(x) = Y^{rs}\delta_{rs} p_i^\mu(x) = Y^{rs}(-M_{rs})_{ij} p_j^\mu(x)$, $\delta q^i = Y^{rs}\delta_{rs} q^i = Y^{rs}(M_{rs})_{ij} q^j$, and $\delta x^\mu(x) = 0$, of the fields, multimomenta, and coordinates of a partial section $\Gamma J^1\mathcal{E}^*$ of the affine dual of the jet bundle is:

$$\delta S_{MP} \;=\; \delta \int_{\Gamma J^1\mathcal{E}^*} \mathcal{L}_{MP}\; d^d x \;=\; \int_{\Gamma J^1\mathcal{E}^*} \delta(p_i^\mu \partial_\mu u^i - \mathcal{H})\; d^d x = 0 \tag{D.9}$$

because $\mathcal{L}_{MP}$ is manifestly a scalar with respect to rotations of the target space coordinates by δ. Expanding the variation of the integrand,

$$\begin{aligned}
\delta S_{MP} \;=\; & \int_{\Gamma J^1\mathcal{E}^*} [\; E_{p_i^\mu}\delta p_i^\mu \;+\; E_{u^i}\delta u^i \;+\; \mathcal{H}_\mu \delta x^\mu \;]\; \mathrm{d}^d x \;+\; \int_{\partial\Gamma J^1\mathcal{E}^*} T_\delta^\mu \mathrm{d}S_\mu \\
=\; & \int_{\Gamma J^1\mathcal{E}^*} [(\partial_\mu q^i - \frac{1}{m}\; g_{\mu\nu}\delta^{ij} p_j^\nu)\delta p_i^\mu \;+\; (-\partial_\nu p_i^\nu - 2V'(\bar{q}^2) q^i)\delta u^i \;+\; \mathcal{H}_\mu \delta x^\mu \\
& \qquad + \partial_\mu(\, \delta u^i p_i^\mu \;+ \mathcal{L}_{MP}\; \delta x^\mu\,)]\; \mathrm{d}^d x \\
=\; & \int_{\Gamma J^1\mathcal{E}^*} [(\partial_\mu q^i - \frac{1}{m}\; g_{\mu\nu}\delta^{ik} p_k^\nu)\; (\; Y^{rs}(-M_{rs})_{ij} p_j^\mu(x)\;) \\
& + (-\partial_\nu p_i^\nu - 2V'(\bar{q}^2) q^i)\; (\; Y^{rs}(M_{rs})_{ij} q^j\;) \;+\; \partial_\mu(p_i^\mu \;\; (\; Y^{rs}(M_{rs})_{ij} q^j\;)\;)\;]\; \mathrm{d}^d x
\end{aligned} \tag{D.10}$$

D.1.3 Conserved current

When the DDW equations are satisfied we obtain

$$\delta\mathcal{L}_{MP} \;= \partial_\mu(\, p_i^\mu \; Y^{rs}(M_{rs})_{ij} q^j\,) \;\;=\;\; Y^{rs}\, \partial_\mu(\, p_i^\mu \; (M_{rs})_{ij} q^j\,) \;\approx 0$$

so we have a conserved current $T_\delta^\mu \;=\; Y^{rs}(\; p_i^\mu (M_{rs})_{ij} q^j\;) \approx\; Y^{rs}(\; m\, \partial_\nu q^i g^{\nu\mu} \delta_{ij} (M_{rs})_{ij} q^j\;)$.

This current generates the variation δ, via this multi-Poisson bracket:

$$\{f, g\}_\mu := \left(\frac{\partial f}{\partial q^i}\right)\left(\frac{\partial g}{\partial p_i^\mu}\right) - \left(\frac{\partial f}{\partial p_i^\mu}\right)\left(\frac{\partial g}{\partial q^i}\right) \tag{D.11}$$

$$\delta u^k \;=\; Y^{rs}\; \delta_{rs} u^k =\; Y^{rs}\{u^k, T_{rs}^\mu\}_\mu =\; Y^{rs}\; \delta_k^l\; \frac{\partial(p_i^\mu (M_{rs})_{ij} q^j)}{\partial p_l^\mu} = (d)\; Y^{rs}\; (M_{rs})_{kj} q^j \tag{D.12}$$

$$\begin{aligned}
\delta p_k^\nu \;=\; & Y^{rs}\; \delta_{rs} p_k^\nu =\; Y^{rs}\{p_k^\nu, T_{rs}^\mu\}_\mu =\; Y^{rs}(\; -\; \delta_\mu^\nu \delta_k^l \frac{\partial(p_i^\mu (M_{rs})_{ij} q^j)}{\partial u^l}\;) \\
& =\; Y^{rs}\; (-M_{rs})_{kj} p_j^\nu
\end{aligned} \tag{D.13}$$

The generators multi-Poisson commute with the DDW Hamiltonian:

$$\{\mathcal{H}, T_{rs}^\mu\}_\mu \;=\; \{\; \frac{1}{2m} p_i^\kappa p_j^\nu\; g_{\kappa\nu}\delta^{ij} \;+ V(\bar{q}^2)\;,\; p_n^\mu (M_{rs})_{nl} q^l\;\}_\mu$$

$$=\; \frac{1}{2m} g_{\kappa\nu}\delta^{ij}\;\; \{\; p_i^\kappa p_j^\nu\;,\; q^l\;\}_\mu\; p_n^\mu (M_{rs})_{nl} \;+\; \{\; V(\bar{q}^2)\;,\; p_n^\mu\;\}_\mu\; (M_{rs})_{nl} q^l$$

$$=\; \frac{1}{2m} g_{\kappa\nu}\delta^{ij}\;\; (\; p_i^\kappa(-\delta_j^m \delta_\mu^\nu \delta_m^l) + (-\delta_i^m \delta_\mu^\kappa \delta_m^l) p_j^\nu\;)\; p_n^\mu (M_{rs})_{nl} + 2V'(\bar{q}^2)\; \{q^2,\; p_n^\mu\;\}_\mu (M_{rs})_{nl} q^l$$

$$= \frac{-1}{2m} g_{\kappa\nu} \; (\delta^{il} \;\; p_i^\kappa p_n^\nu + \delta^{lj} \; p_j^\nu \; p_n^\kappa)(M_{rs})_{nl} \; + \; 2V'(\bar{q}^2) \; q^n (M_{rs})_{nl} q^l$$

$$= \frac{-1}{2m} g_{\kappa\nu} \; (\; p_l^\kappa p_n^\nu + p_l^\nu \; p_n^\kappa)(M_{rs})_{nl} \; + \; 2V'(\bar{q}^2) \; (M_{rs})_{nl} q^n q^l$$

$$= \frac{-1}{2m} g_{\kappa\nu} \; 2 \; (\; p_{(l}^\kappa p_{n)}^\nu)(M_{rs})_{nl} \; + \; 2V'(\bar{q}^2) \; (M_{rs})_{nl} q^{(n} q^{l)} \; = 0 \tag{D.14}$$

This is zero because $g_{\kappa\nu} g_{\kappa\nu}(p_l^\kappa p_n^\nu + p_l^\nu \; p_n^\kappa)$ and $(q^n q^l)$ are symmetric in nl and they contract with $(M_{rs})_{nl}$ which is antisymmetric in nl.

D.1.4 Current algebra

The generators have the same multi-Poisson bracket Lie algebra as $so(N)$:

This is calculated as follows:

$$\{T_{ij}^\nu, T_{kl}^\mu\}_\mu \;\; = \;\; \{ \; p_{n'}^\nu (M_{ij})_{n'l'} \; q^{l'} \;\; , \; p_n^\mu (M_{km})_{nl} \; q^l \; \}_\mu$$

$$= \; p_{n'}^\nu (M_{ij})_{n'l'} \; \{ \; q^{l'} \;\; , \; p_n^\mu\}_\mu \; (M_{km})_{nl} \; q^l \; + \; (M_{ij})_{n'l'} \; q^{l'} \; \{ \; p_{n'}^\nu \; , \; q^l \; \}_\mu \; p_n^\mu (M_{km})_{nl}$$

$$= \; p_{n'}^\nu (M_{ij})_{n'l'} \; \delta_n^{l'}(d) \; (M_{km})_{nl} \; q^l \; + \; (M_{ij})_{n'l'} \; q^{l'} (-\delta_{n'}^l \delta_\mu^\nu) \; p_n^\mu (M_{km})_{nl}$$

$$= \; p_{n'}^\nu (M_{ij})_{n'n} \; (M_{km})_{nl} \; q^l \; - \; p_n^\nu (M_{ij})_{n'l'} \; (M_{km})_{nn'} \; q^{l'}$$

$$= \; p_{n'}^\nu (M_{ij})_{n'n} \; (M_{km})_{nl} \; q^l \; - \; p_{n'}^\nu (M_{ij})_{nl} \; (M_{km})_{n'n} \; q^l$$

$$= \; p_{n'}^\nu \; (\; (M_{ij})_{n'n} \; (M_{km})_{nl} \; - \; (M_{km})_{n'n} \; (M_{ij})_{nl} \;) \; q^l$$

$$= \; p_{n'}^\nu \; (\; (M_{ij})_{n'n} \; (M_{km})_{nl} \; - \; (M_{km})_{n'n} \; (M_{ij})_{nl} \;) \; q^l$$

$$= p_{n'}^\nu \; (\; [\; (M_{ij}), (M_{km}) \;])_{n'l} \; q^l = p_{n'}^\nu \; (\; \delta_{ik} M_{jm} + \delta_{jm} M_{ik} - \delta_{im} M_{jk} - \delta_{jk} M_{im} \;)_{n'l} q^l \tag{D.15}$$

where to obtain the last expression we used the commutator of the M's (D.6). The last expression is $\delta_{ik} T_{jm}^\nu + \; \delta_{jm} T_{ik}^\nu - \; \delta_{im} T_{jk}^\nu - \; \delta_{jk} T_{im}^\nu$ so we have

$$\{T_{ij}^\nu, T_{km}^\mu\}_\mu \;\; = \;\; \delta_{ik} T_{jm}^\nu + \; \delta_{jm} T_{ik}^\nu - \; \delta_{im} T_{jk}^\nu - \; \delta_{jk} T_{im}^\nu \tag{D.16}$$

which is the same algebra as the commutator of the M's (D.6).

It is clear that this generalizes to the case of any matrix Lie algebra with basis $\{(M_{rs})\}$ where the variation of the multiphase-space coordinates is

$$\delta_Y q^i = Y^{rs} \delta_{rs} q^i = Y^{rs} (M_{rs})_{ij} q^j \quad \delta_Y x^\mu = 0 \tag{D.17}$$

$$\delta_Y p_i^\mu(x) = Y^{rs} \delta_{rs} p_i^\mu(x) = Y^{rs} (M_{rs})_{ji} p_j^\mu(x) = Y^{rs} (M_{rs}^T)_{ij} p_j^\mu(x) = -Y^{rs} (M_{rs})_{ij} p_j^\mu(x) \tag{D.18}$$

The current will be $T_\delta^\mu \; = \; Y^{rs}(\; p_i^\mu (M_{rs})_{ij} q^j \;)$ and the current Lie algebra with the multi-Poisson algebra will be the same as the matrix Lie algebra as in the example above.

D.2 The electromagnetic field

D.2.1 Lagrangian analysis

The starting point is the configuration space action of the pure matter free EM field with Lagrangian density $\mathcal{L}$ a function of the electric potential A_0 and the magnetic potential $\vec{A} := (A_1, A_2, A_3)$ and their space time derivatives:

$$S[A_\mu(x)] = \int_{M^d} \mathcal{L}\ \mathrm{d}^d x = \int_{M^d} |\mathrm{d}A|^2\ \mathrm{d}^d x = \int_{M^d} \partial_{[\mu} A_{\nu]}\ \partial_{[\lambda} A_{\rho]}\ g^{\mu\lambda} g^{\nu\rho}\ \mathrm{d}^d x \tag{D.19}$$

where A is a 1-form field on Minkowski spacetime M^d. The Lagrangian density $|\mathrm{d}A|^2$ is off-shell invariant under a local variation $\delta_f A_\rho(x) = \partial_\rho f(x)$, where $f(x)$ is an arbitrary smooth function (locally defined) on spacetime. This is obvious when it is noted that the Lagrangian density depends only on the exterior derivative of A and that $\mathrm{d}(A + \mathrm{d}f) = \mathrm{d}A + \mathrm{d}\mathrm{d}f = \mathrm{d}A + 0$, so it is dependent on A up to addition with a closed 1-form. The associated current is $J_f^\mu = F^{\mu\rho}\partial_\rho f(x)$ where $\frac{\partial \mathcal{L}}{\partial(\partial_\nu A_\mu)} = F^{\mu\nu} = F^{[\mu\nu]} := 2\partial_{[\lambda} A_{\rho]}\ g^{\mu\lambda} g^{\nu\rho}$.

The Euler-Lagrange equations are $\partial_\nu F^{\mu\nu} \approx 0$, which are the source free Maxwell's equations in terms of second spacetime derivatives of the potentials A_μ.

The divergence of the current is $\partial_\mu J_f^\mu = \partial_\mu F^{\mu\rho}\partial_\rho f + F^{\mu\rho}\partial_\mu\partial_\rho f(x) = \partial_\mu F^{\mu\rho}\partial_\rho f \approx 0$, where we made use of the antisymmetry of $F^{\mu\nu}$ and the symmetry of partial derivatives. Thus the current is conserved on-shell.

If an interaction term $A_\mu J_\phi^\mu$ is added to the Lagrangian density $\mathcal{L}$, where J_ϕ^μ is a function of other matter fields ϕ but not A, the Euler-Lagrange equations are $\partial_\nu F^{\mu\nu} \approx J_\phi^\mu$.

The divergence of the current with the interaction term present is $\partial_\mu J_f^\mu = \partial_\mu F^{\mu\rho}\partial_\rho f + F^{\mu\rho}\partial_\mu\partial_\rho f(x) = \partial_\mu F^{\mu\rho}\partial_\rho f \approx J_\phi^\rho \partial_\rho f$. But

$$\delta_f S[A_\mu(x)] \approx \int_{M^d} \partial_\mu J_f^\mu \mathrm{d}^d x = \int_{M^d} J_\phi^\rho \partial_\rho f \mathrm{d}^d x = -\int_{M^d} \partial_\rho J_\phi^\rho f \mathrm{d}^d x \tag{D.20}$$

ignoring the boundary term. Thus the current is conserved and the action is gauge invariant on-shell if the matter current is also conserved: $\partial_\rho J_\phi^\rho \approx 0$.

D.2.2 Legendre transformation and Hamiltonian analysis

The multiphase-space Legendre transformation maps the jet bundle to multiphase space with multimomenta $p^{\mu\nu} \approx 2\partial_{[\lambda} A_{\rho]}\ g^{\mu\lambda} g^{\nu\rho} =: F^{\mu\nu}$ canonically conjugate to A_ρ. There are primary constraints $p^{(\mu\nu)} \approx 0$ (but not secondary constraints). The DDW Hamiltonian is

$$\mathcal{H} = p^{\mu\nu}\partial_\mu A_\nu \ - L = \frac{1}{4}p^{[\mu\nu]}p_{[\mu\nu]} + p^{(\mu\nu)}\partial_\mu A_\nu \ = \frac{1}{4}p^{[\mu\nu]}p_{[\mu\nu]} - \partial_\mu p^{(\mu\nu)}A_\nu \tag{D.21}$$

employing integration by parts inside the first order action (D.26) for the last equality. The term with spacetime partial derivatives is usual when there is a primary constraint, because not all the velocities can be replaced by momenta, because of the non-invertibility of the Legendre transformation.

The DeDonder Weyl equations of motion are:

$$\begin{aligned} \partial_\mu A_\nu(x) - \{A_\nu, \mathcal{H}\}_\mu = \partial_\mu A_\nu(x) - \frac{\partial \mathcal{H}}{\partial p^{\mu\nu}}(x, A_\kappa(x), p^{\lambda\kappa}(x)) \approx 0 \\ \partial_\mu p^{\mu\nu}(x) - \{p^{\mu\nu}, \mathcal{H}\}_\mu = \partial_\mu p^{\mu\nu}(x) + \frac{\partial \mathcal{H}}{\partial A_\nu}(x, A_\kappa(x), p^{\lambda\kappa}(x)) \approx 0 \end{aligned} \tag{D.22}$$

which are here, substituting for $\mathcal{H}$,

$$\begin{aligned} \partial_\mu A_\nu(x) - \frac{1}{2}p_{[\mu\nu]}(x) - \partial_{(\mu}A_{\nu)}(x) \approx 0 \\ \partial_\mu p^{\mu\nu}(x) - \partial_\mu p^{(\mu\nu)}(x) \approx 0 \end{aligned} \tag{D.23}$$

this can be written

$$\begin{aligned} \partial_{[\mu}A_{\nu]}(x) - \frac{1}{2}p_{[\mu\nu]}(x) = 0 \\ \partial_\mu p^{[\mu\nu]}(x) = 0 \end{aligned} \tag{D.24}$$

By taking the partial derivative ∂_λ of the first line above and antisymmetrizing , because $\partial_{[\lambda}\partial_\mu A_{\nu]} = (\mathrm{dd}A)_{\lambda\mu\nu} = 0$, we can eliminate the A and obtain equations of motion purely in terms of spacetime partial derivatives of the antisymmetric part of the multimomenta:

$$\begin{aligned} \partial_{[\lambda}p_{\mu\nu]}(x) = 0 \\ \partial_\mu p^{[\mu\nu]}(x) = 0 \end{aligned} \tag{D.25}$$

which are the source free Maxwell's equations, where $p_{[\mu\nu]} \approx F_{\mu\nu}$, $p_{[0i]} \approx F_{0i} = E_i$ and $\epsilon_{ijk}p_{[jk]} \approx \epsilon_{ijk}F_{jk} = B_i$ in the conventional notation for the electromagnetic field.

These are the Euler-Lagrange equations for a stationary point of the following action:

$$\begin{aligned} S_{MP} = \int_{\Gamma J^1\mathcal{E}^*}(p^{\mu\nu}\partial_\mu A_\nu - \mathcal{H})d^dx = \int_{\Gamma J^1\mathcal{E}^*}(p^{\mu\nu}\partial_\mu A_\nu - \frac{1}{4}p^{[\mu\nu]}p_{[\mu\nu]} - p^{(\mu\nu)}\partial_\mu A_\nu \ )d^dx \\ = \int_{\Gamma J^1\mathcal{E}^*}(p^{[\mu\nu]}\partial_\mu A_\nu - \frac{1}{4}p^{[\mu\nu]}p_{[\mu\nu]})d^dx \end{aligned} \tag{D.26}$$

which has variation

$$\delta S_{MP} = \delta\int_{\Gamma J^1\mathcal{E}^*}(p^{\mu\nu}\partial_\mu A_\nu - \mathcal{H})d^dx =$$

$$\int_{\Gamma J^1\mathcal{E}^*}(\partial_\mu A_\nu - \frac{\partial \mathcal{H}}{\partial p^{\mu\nu}})\delta p^{\mu\nu} - (\partial_\mu p^{\mu\nu} + \frac{\partial \mathcal{H}}{\partial A_\nu})\delta A_\nu \ + \ \partial_\mu[\delta A_\nu p^{\mu\nu}]d^dx =$$

$$\int_{\Gamma J^1\mathcal{E}^*} (\partial_\mu A_\nu(x) - \frac{1}{2}p_{[\mu\nu]}(x) - \partial_{(\mu}A_{\nu)}(x))\delta p^{\mu\nu} - (\partial_\mu p^{\mu\nu}(x) - \partial_\mu p^{(\mu\nu)}(x))\delta A_\nu \ d^d x$$
$$+ \int_{\partial\Gamma J^1\mathcal{E}^*} [\,\delta A_\nu p^{\mu\nu}\,]\mathrm{d}S_\mu \tag{D.27}$$

It can be seen from the last line that extremizing the integral with A_μ fixed on the spacetime boundary gives the Euler-Lagrange equations of motion which are the DDW equations (D.23) above.

D.2.3 The multiphase-space energy momentum tensor

The multiphase-space energy momentum tensor density defined in (3.60), is, for the electromagnetism phase space:

$$\tilde{T^\mu_{\ \nu}} := \frac{\partial \mathcal{H}}{\partial p^{\kappa\nu}}p^{\kappa\mu} - (p^{\kappa\lambda}\frac{\partial \mathcal{H}}{\partial p^{\kappa\lambda}} - \mathcal{H})\delta^\mu_\nu \tag{D.28}$$

Substituting the DDW Hamiltonian for the electromagnetic field above, the energy-momentum tensor is then:

$$\tilde{T^\mu_{\ \nu}} := \frac{1}{2}p^{\kappa\mu}p_{[\kappa\nu]} + p^{\kappa\mu}\partial_{(\kappa}A_{\nu)} \ - \ \frac{1}{4}p^{[\kappa\lambda]}p_{[\kappa\lambda]}\delta^\mu_\nu \tag{D.29}$$

D.2.4 Constraints as generators of gauge variations

The generator $f(x)\partial_\mu p^{\mu\nu}\mathrm{d}x_\nu = 0$ generate the gauge transformations $\delta_f A_\rho = \partial_\rho f(x)$, $\delta_f p^{\mu\nu} = 0$, under which the original Lagrangian (D.19) and the first order Lagrangian (D.26) are invariant, via the multi-bracket:

$$-\delta_f A_\rho = \{\,-f\partial_\mu p^{\mu\alpha}\mathrm{d}x_\alpha, A_\rho\} = \frac{1}{d}\{\,-f\partial_\mu p^{\mu\alpha}, A_\rho\}_\alpha = -\frac{1}{d}\mathrm{d}(\,-f\partial_\mu p^{\mu\alpha})\llcorner\Pi_\alpha\lrcorner\mathrm{d}(A_\rho)$$
$$= \ -\frac{1}{d}f\partial_\mu p^{\mu\alpha}\cdot(\frac{\overleftarrow{\partial}}{\partial A_\kappa}\wedge\frac{\overrightarrow{\partial}}{\partial p^{\kappa\alpha}})\cdot A_\rho = \frac{1}{d}\partial_\mu f\ p^{\mu\alpha}\cdot(\frac{\overleftarrow{\partial}}{\partial A_\kappa}\wedge\frac{\overrightarrow{\partial}}{\partial p^{\kappa\alpha}})\cdot A_\rho$$
$$= -\frac{1}{d}\partial_\mu f\left(\frac{\partial p^{\mu\alpha}}{\partial p^{\mu\alpha}}\right)\left(\frac{\partial A_\rho}{\partial A_\kappa}\right) = -\frac{1}{d}\partial_\mu f(\delta^\mu_\kappa\delta^\alpha_\alpha)\delta^\kappa_\rho = -\partial_\rho f \tag{D.30}$$

D.2.5 Multisymplectic analysis

The gauge variation of the multimomenta $p^{\nu\beta}$ is

$$-\delta_f p^{\nu\beta} = \{\,-f\partial_\mu p^{\mu\alpha}\mathrm{d}x_\alpha, p^{\nu\beta}\} = \frac{1}{d}\{\,\partial_\mu f p^{\mu\alpha}, p^{\nu\beta}\}_\alpha = 0 \tag{D.31}$$

The multi-brackets of the constraint with the DDW Hamiltonian is:

$$\{\mathcal{H}, p^{(\mu\alpha)}\}_\alpha = \mathcal{H}\cdot(\frac{\overleftarrow{\partial}}{\partial A_\kappa}\wedge\frac{\overrightarrow{\partial}}{\partial p^{\kappa\alpha}})\cdot p^{(\mu\alpha)} = -\partial_\lambda p^{(\lambda\kappa)}\frac{1}{2}(\delta^\mu_\kappa\delta^\alpha_\alpha+\delta^\alpha_\kappa\delta^\mu_\alpha) = -\partial_\lambda p^{(\lambda\mu)}\frac{1}{2}(d+1) \quad \text{(D.32)}$$

which is zero on the constraint surface $p^{(\lambda\kappa)} = 0$, which makes $p^{(\mu\nu)}, \mathcal{H}$ first class constraints, which are also abelian:

$$\{p^{(\mu\alpha)}, p^{(\nu\beta)}\}_\gamma = 0 \quad \text{(D.33)}$$

The current produced by a gauge variation is

$$J_f = (\partial_\alpha f)p^{\alpha\mu}\mathrm{d}x_\mu \quad \text{(D.34)}$$

The current generates the gauge variation via the multibracket:

$$X_{J_f} = \{\cdot, (\partial_\alpha f)p^{\alpha\mu}\mathrm{d}x_\mu\} = \{\cdot, \partial_\alpha f p^{(\alpha\mu)}\mathrm{d}x_\mu\} = (\partial_\alpha f)\frac{\partial}{\partial A_\alpha} \quad \text{(D.35)}$$

so it has one of the attributes of a hamiltonian $d-1$ -form corresponding to the gauge variation.

To check the hamiltonian property, $\mathrm{d}J_f = X_{J_f} \lrcorner \Omega$:

$$\mathrm{d}J_f = (\partial_\alpha f)\mathrm{d}p^{\alpha\mu}\wedge\mathrm{d}x_\mu + (\partial_\beta\partial_\alpha f)p^{\alpha\mu}\mathrm{d}x^\beta\wedge\mathrm{d}x_\mu = (\partial_\alpha f)\mathrm{d}p^{\alpha\mu}\wedge\mathrm{d}x_\mu + (\partial_\beta\partial_\alpha f)p^{(\alpha\beta)}\mathrm{d}^d x$$

$$\stackrel{c}{=} (\partial_\alpha f)\mathrm{d}p^{[\alpha\mu]}\wedge\mathrm{d}x_\mu \quad \text{(D.36)}$$

$$X_{J_f}\lrcorner\Omega = X_{J_f}\lrcorner\mathrm{d}A_\alpha\wedge\mathrm{d}p^{\alpha\mu}\wedge\mathrm{d}x_\mu = (\partial_\alpha f)\mathrm{d}p^{\alpha\mu}\wedge\mathrm{d}x_\mu \stackrel{c}{=} (\partial_\alpha f)\mathrm{d}p^{[\alpha\mu]}\wedge\mathrm{d}x_\mu \quad \text{(D.37)}$$

It can be seen that $\mathrm{d}J_f = X_{J_f}\lrcorner\Omega$ on the constraint surface $p^{(\alpha\beta)} = 0$, so the gauge variation is hamiltonian on the constraint surface, indicated by the notation "$\stackrel{c}{=}$".

The Lie derivative of the canonical d form with respect to the gauge variation is:

$$\mathcal{L}_{X_{J_f}}\Theta = X_{J_f}\lrcorner\mathrm{d}\Theta + \mathrm{d}(X_{J_f}\lrcorner\Theta)$$

$$= -(\partial_\alpha f)\frac{\partial}{\partial A_\alpha}\lrcorner\mathrm{d}A_\alpha\wedge\mathrm{d}p^{\alpha\mu}\wedge\mathrm{d}x_\mu + \mathrm{d}(\partial_\alpha f)\frac{\partial}{\partial A_\alpha}\lrcorner(p^{\alpha\mu}\mathrm{d}A_\alpha\wedge\mathrm{d}x_\mu)$$

$$= (\partial_\alpha\partial_\mu f)\mathrm{d}p^{(\alpha\mu)}\wedge\mathrm{d}^d x \stackrel{c}{=} 0 \quad \text{(D.38)}$$

So the gauge variation is exact on the constraint surface.

D.2.6 MW reduction: constraint surface

The DDW Hamiltonian (given by the Legendre transformation) is

$$\mathcal{H} = \frac{1}{4}p^{[\mu\nu]}p_{[\mu\nu]} - \partial_\mu p^{(\mu\nu)}A_\nu \stackrel{c}{=} \frac{1}{4}p^{[\mu\nu]}p_{[\mu\nu]} \approx \frac{1}{2}(E^2+B^2) \quad \text{(D.39)}$$

on the constraint surface $p^{(\alpha\beta)} = 0$ in multiphase space.

The gauge variation of $\mathcal{H}$ is

$$\delta\mathcal{H} = \partial_\mu p^{(\mu\nu)}\delta A_\nu \;=\; \partial_\mu p^{(\mu\nu)}\partial_\nu f(x) \;\stackrel{c}{=}\; 0 \tag{D.40}$$

For gauge invariance of the DDW Hamiltonian we must restrict to the constraint surface $p^{(\alpha\beta)} = 0$.

The Lie algebra $\mathfrak{g}$ is $C^\infty(M^d)/\mathbb{R}$ and abelian. The moment map is $\mathcal{M} \longrightarrow \mathfrak{g}* :: m = (A_\alpha, p^{\alpha\mu}, x) \mapsto \mathfrak{I}(m) = p^{(\alpha\mu)}\mathrm{d}x_\mu$

Elements of the Lie algebra $f(x)$ are best viewed as vectors $(\partial_\alpha f)_{\alpha=0\ldots d-1}$: For $f(x) \in \mathfrak{g}$, $\langle\mathfrak{I}(m), f\rangle = (\partial_\alpha f)p^{(\alpha\mu)}\mathrm{d}x_\mu$.

The moment map is zero on the surface $p^{(\alpha\beta)} = 0$. The constraint surface is $M_G = \{(A_\alpha, p^{[\alpha\beta]}, x)\}$. Following the lead of symplectic Marsden-Weinstein reduction, we want to show that $Y \lrcorner \Omega_G = Y \lrcorner \mathrm{d}A_\alpha \wedge \mathrm{d}p^{[\alpha\mu]} \wedge \mathrm{d}x_\mu = 0$ implies $Y = (\partial_\alpha f)\frac{\partial}{\partial A_\alpha}$, the infinitesimal gauge variation vector field. This would lead to the second stage in the generalized Marsden-Weinstein reduction which is to mod-out the characteristic distribution from the tangent space of the constraint submanifold M_G.

The Lie derivative of the multisymplectic form with respect to an arbitrary infinitesimal variation $\delta A_\alpha = B_\alpha$ in A is:

$$\begin{aligned}\mathcal{L}_{X_\delta}\Omega &= X_\delta \lrcorner \mathrm{d}\Omega + \mathrm{d}(X_\delta \lrcorner \Omega)\\ &= 0 + \mathrm{d}(B_\alpha(x))\frac{\partial}{\partial A_\alpha} \lrcorner \mathrm{d}A_\alpha \wedge \mathrm{d}p^{\alpha\mu} \wedge \mathrm{d}x_\mu = (-\partial_\mu B_\alpha)\ \mathrm{d}p^{\alpha\mu} \wedge \mathrm{d}^d x\end{aligned} \tag{D.41}$$

This is not zero unless $B_\alpha =$ constant, so this is not a multisymplectomorphism over multiphase space. If we restrict the multisymplectic form to the constraint surface $p^{(\alpha\beta)} = 0$ in multiphase space, we have

$$\mathcal{L}_{X_\delta}\Omega \stackrel{c}{=} (\partial_\mu B_\alpha)\ \mathrm{d}p^{[\alpha\mu]} \wedge \mathrm{d}^d x \tag{D.42}$$

This is zero if $\partial_{[\mu}B_{\alpha]}(x) = 0$. Viewing B_α as the components of a 1-form, this implies, by Darboux's theorem, that $B_\alpha = \partial_\alpha f(x)$ for any function $f(x)$ locally defined on spacetime (in fact, on flat space, $f(x)$ is defined on all the space). So the form of the gauge variation, $\delta A_\alpha = \partial_\alpha f(x)$ is set by the requirement that the variation is a multisymplectomorphism of the multisymplectic form restricted to the constraint surface $p^{(\alpha\beta)} = 0$.

The Lie derivative of the canonical d form with respect to an arbitrary infinitesimal variation in A is:

$$\begin{aligned}\mathcal{L}_{X_\delta}\Theta &= X_\delta \lrcorner \mathrm{d}\Theta + \mathrm{d}(X_\delta \lrcorner \Theta)\\ &= -(B_\alpha)\frac{\partial}{\partial A_\alpha} \lrcorner \mathrm{d}A_\alpha \wedge \mathrm{d}p^{\alpha\mu} \wedge \mathrm{d}x_\mu + \mathrm{d}(B_\alpha)\frac{\partial}{\partial A_\alpha} \lrcorner (p^{\alpha\mu}\mathrm{d}A_\alpha \wedge \mathrm{d}x_\mu)\\ &= (\partial_\mu B_\alpha)\mathrm{d}p^{\alpha\mu} \wedge \mathrm{d}^d x \stackrel{c}{=} (\partial_{[\mu}B_{\alpha]})\mathrm{d}p^{[\alpha\mu]} \wedge \mathrm{d}^d x\end{aligned} \tag{D.43}$$

So the gauge variation, $\delta A_\alpha = \partial_\alpha f(x)$ which leads to a multisymplectomorphism of the multisymplectic form when restricted to the constraint surface $p^{(\alpha\beta)} = 0$ is also exact.

D.2.7 MW reduction: foliation by the symmetry action

These conditions are not as strong as the condition $X_\delta \lrcorner \Omega_G \overset{c}{=} 0$ for the characteristic vector fields which are used for the second stage of *symplectic* reduction. In symplectic reduction, the set of characteristic vector fields are integrable into leaves of a foliation of the constraint surface because they obey the Frobenius condition (which is a consequence from the Poisson property of the characteristic vector fields), and so the reduced phase space is the space of leaves of the integrated characteristic tangent space in the constraint surface. In terms of functions on phase space the physical observables are the functions on the constraint surface mod the functions on the integrated characteristic tangent space. In the multisymplectic case above the second stage of multisymplectic reduction is that the physical observables are the functions of $p^{[\alpha\mu]}$ and A_α on the constraint surface where A_α is defined mod the functions B_α for which $\partial_{[\mu}B_{\alpha]}(x) = 0$. More succinctly $A = A_\alpha \mathrm{d}x^\alpha$ is physical up to addition with closed forms. The closed form condition is a differential condition in that it does not simply mod out submanifolds of the constraint surface in multiphase space as in the symplectic case. However, by certain types of gauge fixing, the functional reduction to physical phase space can be implemented by reducing the constraint submanifold directly. We will now show an example of this.

We assume the spacetime dimension $d = 4$ and choose the temporal gauge $A_0 = 0$, which is a pointwise constraint in multiphase space unlike the Lorenz or Coulomb gauges which are constraints which are spacetime partial derivatives of A. Then the reduced multiphase space is $M//G = \{(A_i, p^{[ij]}, p^{[i0]}, x)\}$ and the multisymplectic form on this space is

$$\Omega_{GG} = \mathrm{d}A_i \wedge (\mathrm{d}p^{[ij]} \wedge \mathrm{d}x_j + \mathrm{d}p^{[i0]} \wedge \mathrm{d}x_0) = \mathrm{d}A_i \wedge (\mathrm{d}(\epsilon^{ijk}B_k) \wedge \mathrm{d}x_j + \mathrm{d}E^i \wedge \mathrm{d}x_0) \tag{D.44}$$

The DDW Hamiltonian on the reduced multiphase space is

$$\mathcal{H} = \frac{1}{4}(p^{[ij]}p_{[ij]} + p^{i0}p_{i0}) - A_iJ^i = \frac{1}{2}(B^2 + E^2) - A_iJ^i \tag{D.45}$$

The DeDonder Weyl equations of motion are:

$$\begin{aligned}
\partial_\nu A_i(x) - \{A_i, \mathcal{H}\}_\nu &= \partial_\nu A_i(x) - \frac{\partial \mathcal{H}}{\partial p^{i\nu}}(x, A_j(x), p^{j\kappa}(x)) \approx 0 \\
\partial_\nu p^{i\nu}(x) - \{p^{i\nu}, \mathcal{H}\}_\nu &= \partial_\nu p^{i\nu}(x) + \frac{\partial \mathcal{H}}{\partial A_i}(x, A_j(x), p^{j\kappa}(x)) \approx 0
\end{aligned} \tag{D.46}$$

which are here, substituting for $\mathcal{H}$,

$$\partial_\nu A_i(x) - \frac{1}{2}p_{i\nu}(x) \approx 0$$

$$\partial_\nu p^{i\nu}(x) + J^i \approx 0 \tag{D.47}$$

using E and B notation this is:

$$\begin{aligned} \partial_0 A_i(x) - \frac{1}{2} E_i(x) &\approx 0 \\ \partial_j A_i(x) - \frac{1}{2}\epsilon_{ijk} B^k(x) &\approx 0 \\ \partial_0 E^i + \epsilon^{ijk}\partial_j B_k(x) + J^i &\approx 0 \end{aligned} \tag{D.48}$$

which are (after eliminating A_i) the Maxwell equations. A similar multisymplectic analysis of Yang-Mills is in section 3.8.

D.3 The bosonic string

A d dimensional p-brane B (with $p = d-1$) embedded in n dimensional spacetime $(\mathbb{M}, g)$, with metric g_{ij}, can be described by a map $X : B \longrightarrow \mathbb{M}$ where X maps the point x in the brane B (with local coordinates $x^\alpha, \alpha = 0, \ldots, d-1$) to the point $X(x)$ in spacetime $\mathbb{M}$ (with local coordinates $X^i, i = 0, \ldots, n-1$): $X : (\sigma^\alpha) \mapsto (X^i)$. The dynamical action is the covariant volume of the embedding in spacetime. The tension of the p-brane is a constant T.

The starting point is the Nambu-Goto action for a d dimensional p-brane B embedded in n dimensional spacetime [90] expressed in the conventional notation:

$$S_{NG}[X^i(x)] = \int_B \mathcal{L}_{NG}\ \mathrm{d}^d x \ = \ - T \int_B \frac{1}{2}\sqrt{-\det(\partial_\alpha X^i \partial_\beta X^j g_{ij}(X))}\ \mathrm{d}^d x \tag{D.49}$$

The integral is the covariant volume of the embedding in spacetime.

Because of the difficulties created by the square root in the Lagrangian density, the following Howe-Tucker action [73] [81], which has auxiliary dynamical variables $h^{\alpha\beta}(x)$, is used instead.

$$\begin{aligned} S_{HT}[X^i(x), h^{\alpha\beta}(x)] = \int_B \mathcal{L}_{HT}\ \mathrm{d}^d x \ = \\ - T \int_B \sqrt{-\det(h_{\alpha\beta})}[h^{\alpha\beta}\partial_\alpha X^i \partial_\beta X^j g_{ij}(X) - (d-2)]\ \mathrm{d}^d x \end{aligned} \tag{D.50}$$

This action is classically equivalent to the Nambu-Goto action in that, when the auxiliary world-volume metric $h^{\alpha\beta}(x)$ is solved for, using the Euler-Lagrange equations, and substituted back into the Howe-Tucker action, the result is the Nambu-Goto action.

The multiphase-space Legendre transformation gives

$$p_i^\alpha \approx \frac{\partial \mathcal{L}_{HT}}{\partial(\partial_\alpha X^i)} = \ -2Th^{\frac{1}{2}}\ h^{\alpha\beta}\partial_\beta X^j g_{ij} \quad \text{and} \quad H^\gamma_{\alpha\beta} \approx \frac{\partial \mathcal{L}_{HT}}{\partial(\partial_\gamma h_{\alpha\beta})} = 0 \tag{D.51}$$

where $h := \sqrt{-\det(h_{\alpha\beta})}$.

D.3.1 DDW Hamiltonian

From this we obtain

$$\partial_\alpha X^i \approx -\frac{1}{2}T^{-1}h^{-\frac{1}{2}}h_{\alpha\beta}p_j^\beta g^{ij} \quad \text{and} \quad \mathcal{H} = -[\frac{1}{4}T^{-1}h^{-\frac{1}{2}}h_{\alpha\beta}p_i^\alpha p_j^\beta g^{ij} + Th^{\frac{1}{2}}(d-2) - H^{\gamma\alpha\beta}\partial_\gamma h_{\alpha\beta}] \tag{D.52}$$

where $\mathcal{H}$ is the DDW Hamiltonian.

The HT action is equivalent to stationary point of the following first order action:

$$\begin{aligned} S_{MP} = & \int_{\Gamma J^1\mathcal{E}^*} (p_i^\alpha \partial_\alpha X^i + H^{\gamma\alpha\beta}\partial_\gamma h_{\alpha\beta} - \mathcal{H}) d^d x \ = \\ = & \int_{\Gamma J^1\mathcal{E}^*} (p_i^\alpha \partial_\alpha X^i + H^{\gamma\alpha\beta}\partial_\gamma h_{\alpha\beta} \\ & + [\ \frac{1}{4}T^{-1}h^{-\frac{1}{2}}h_{\alpha\beta}p_i^\alpha p_j^\beta g^{ij} + Th^{\frac{1}{2}}(d-2) - H^{\gamma\alpha\beta}\partial_\gamma h_{\alpha\beta}])d^d x \ = \\ = & \int_{\Gamma J^1\mathcal{E}^*} (p_i^\alpha \partial_\alpha X^i + [\frac{1}{4}T^{-1}h^{-\frac{1}{2}}h_{\alpha\beta}p_i^\alpha p_j^\beta g^{ij} + Th^{\frac{1}{2}}(d-2)])d^d x \end{aligned} \tag{D.53}$$

The Euler-Lagrange equations, which are also the DeDonder Weyl equations of motion, are:

$$E_{p_i^\alpha} \ \equiv \ \partial_\alpha X^i(x) - \frac{\partial \mathcal{H}}{\partial p_i^\alpha}(X^j(x), p_j^\beta(x), h_{\alpha\beta}) \ \approx \ 0 \qquad \text{(DDW1)} \tag{D.54}$$

$$-E_{X^i} \ \equiv \ \partial_\alpha p_i^\alpha \ + \ \frac{\partial \mathcal{H}}{\partial X^i}(X^j(x), p_j^\beta(x), h_{\alpha\beta}) \ \approx \ 0 \qquad \text{(DDW2)} \tag{D.55}$$

$$-E_{h_{\alpha\beta}} \ \equiv \ \partial_\gamma H^{\gamma\alpha\beta} + \frac{\partial \mathcal{H}}{\partial h_{\alpha\beta}}(X^j(x), p_j^\beta(x), h_{\alpha\beta}) \ \approx \ 0 \qquad \text{(DDW2)} \tag{D.56}$$

$$E_{H^{\gamma\alpha\beta}} \ \equiv \ \partial_\gamma h_{\alpha\beta} - \frac{\partial \mathcal{H}}{\partial H^{\gamma\alpha\beta}}(X^j(x), p_j^\beta(x), h_{\alpha\beta}) = \partial_\gamma h_{\alpha\beta} - \partial_\gamma h_{\alpha\beta} \ \approx \ 0 \qquad \text{(DDW1)} \tag{D.57}$$

which are, substituting for $\mathcal{H}$ above,

$$\begin{aligned} E_{p_i^\alpha} \ &\equiv \ \partial_\alpha X^i(x) + \frac{1}{2}T^{-1}h^{-\frac{1}{2}}h_{\alpha\beta}p_j^\beta g^{ij} \ \approx \ 0 \\ -E_{X^i} \ &\equiv \ \partial_\alpha p_i^\alpha(x) - \frac{1}{4}T^{-1}h^{-\frac{1}{2}}h_{\alpha\beta}p_i^\alpha p_j^\beta \frac{\partial g^{ij}}{\partial X^i} \ \approx \ 0 \\ -E_{h_{\alpha\beta}} \ \equiv \ T^{\alpha\beta} &= -\frac{1}{4}T^{-1}h^{-\frac{1}{2}}\ [\ \delta^\alpha_{\alpha'}\delta^\beta_{\beta'} - \frac{1}{2}h^{\alpha\beta}h_{\alpha'\beta'}\ ]\ p_i^{\alpha'}p_j^{\beta'}g^{ij} + \frac{1}{2}Th^{\alpha\beta}h^{\frac{1}{2}}(d-2) \ \approx \ 0 \\ E_{H^{\gamma\alpha\beta}} \ &\equiv \ \partial_\gamma h_{\alpha\beta} - \partial_\gamma h_{\alpha\beta} \ \approx \ 0 \end{aligned} \tag{D.58}$$

where $T^{\alpha\beta}$ is the energy-momentum tensor density. The last line shows that the world sheet metric $h_{\alpha\beta}$ are not independent physical degrees of freedom. This is directly connected with the constraint $T^{\alpha\beta} \approx 0$ in the 3rd line and the fact that $H^{\gamma\alpha\beta}$, the multimomentum canonically conjugate to $h_{\alpha\beta}$, is not present in the equations of motion or in the Lagrangian, and so decouples from the other dynamical variables.

D.3.2 Primary constraint

In the case of a 1-brane (string), where $p+1=d=2$, the equation for $h_{\alpha\beta}$ simplifies to :

$$-E_{h_{\alpha\beta}} \;=\; T^{\alpha\beta} = \;-\frac{1}{4}T^{-1}h^{-\frac{1}{2}}\;(\;p_i^\alpha p_j^\beta - \frac{1}{2}h^{\alpha\beta}h_{\alpha'\beta'}p_i^{\alpha'}p_j^{\beta'})g^{ij}\;\approx\;0 \tag{D.59}$$

This is the primary constraint corresponding to the gauge degrees of freedom of $h_{\alpha\beta}$ corresponding to the invariance under change of coordinates on the string: $\delta_f h_{\alpha\beta} = \partial_{(\alpha}f_{\beta)}$ where $f_\beta(\sigma)\mathrm{d}x^\beta$ is an arbitrary 1-form.

Because the DDW Hamiltonian is not linear but quadratic in the 'Lagrange multiplier' $h_{\alpha\beta}$, the constraint is not on the other phase-space coordinates, but on $h_{\alpha\beta}$ which is now a function of the other phase-space coordinates.

The constraint has solution:

$$h^{\alpha\beta}\;\approx\;e^{2\phi(x)}p_i^\alpha p_j^\beta g^{ij} \tag{D.60}$$

where $\phi(x)$ is arbitrary, so, on shell, the worldsheet metric $h^{\alpha\beta}(x)$ must be proportional to the metric induced on the worldsheet by the spacetime metric g^{ij} via the 'embedding' of multimomenta: $g^{ij}(X(x))p_i^\alpha p_j^\beta(x) \approx h^{\alpha\beta}\partial_\alpha X^i\partial_\beta X^j g_{ij}(X)$.

The gauge freedom, in the case $d=2$, represented by the arbitrary $\phi(x)$ is connected to the fact that, in the case $d=2$ only, the energy-momentum tensor is traceless:

$$T^\alpha{}_\alpha = T^{\alpha\beta}h_{\alpha\beta} = -\frac{1}{4}T^{-1}h^{-\frac{1}{2}}\;[\;h_{\alpha'\beta'} - \frac{1}{2}(d)h_{\alpha'\beta'}\;]\;p_i^{\alpha'}p_j^{\beta'}g^{ij} + \frac{1}{2}Th^{\alpha\beta}h^{\frac{1}{2}}(d-2) \tag{D.61}$$

Both terms are zero when $d=2$. Thus the trace of the energy-momentum tensor is zero for the string.

The multi-Poisson brackets here are

$$\{f,g\}_\gamma := -\mathrm{d}_v f \llcorner \Pi_\gamma \lrcorner\; \mathrm{d}_v g = f\cdot(\frac{\overleftarrow{\partial}}{\partial X^i}\wedge\frac{\overrightarrow{\partial}}{\partial p_i^\gamma})\cdot g + f\cdot(\frac{\overleftarrow{\partial}}{\partial h_{\alpha\beta}}\wedge\frac{\overrightarrow{\partial}}{\partial H^{\gamma\alpha\beta}})\cdot g =$$

$$\left(\frac{\partial f}{\partial X^i}\right)\left(\frac{\partial g}{\partial p_i^\gamma}\right) - \left(\frac{\partial f}{\partial p_i^\gamma}\right)\left(\frac{\partial g}{\partial X^i}\right) + \left(\frac{\partial f}{\partial h_{\alpha\beta}}\right)\left(\frac{\partial g}{\partial H^{\gamma\alpha\beta}}\right) - \left(\frac{\partial f}{\partial H^{\gamma\alpha\beta}}\right)\left(\frac{\partial g}{\partial h_{\alpha\beta}}\right) \tag{D.62}$$

D.4 General Relativity

Starting from the Einstein-Hilbert action of general relativity [65] :

$$S_{EH}[g_{\mu\nu}(x)] = \int_B \mathcal{L}_{EH}\;\mathrm{d}^d x \;=\; \frac{1}{2\kappa}\int_B g^{\alpha\beta}R_{\alpha\beta}(g^{\mu\nu},\partial_\kappa g_{\mu\nu})\sqrt{-\det(g_{\mu\nu})}\;\mathrm{d}^d x =$$

$$\frac{1}{2\kappa}\int_B \bar{g}^{\alpha\beta}R_{\alpha\beta}(\Gamma^\lambda_{\mu\nu},\partial_\kappa\Gamma^\lambda_{\mu\nu})\;\mathrm{d}^d x = \frac{1}{2\kappa}\int_B \bar{g}^{\alpha\beta}(\partial_\kappa\Gamma^\kappa_{\alpha\beta} - \partial_\alpha\Gamma^\kappa_{\kappa\beta} + \Gamma^\kappa_{\kappa\lambda}\Gamma^\lambda_{\alpha\beta} - \Gamma^\kappa_{\alpha\lambda}\Gamma^\lambda_{\kappa\beta})\;\mathrm{d}^d x =$$

$$\frac{1}{2\kappa}\int_B[-\partial_\kappa\bar{g}^{\alpha\beta}\Gamma^\kappa_{\alpha\beta}+\partial_\alpha\bar{g}^{\alpha\beta}\Gamma^\kappa_{\kappa\beta}+\bar{g}^{\alpha\beta}\Gamma^\kappa_{\kappa\lambda}\Gamma^\lambda_{\alpha\beta}-\bar{g}^{\alpha\beta}\Gamma^\kappa_{\alpha\lambda}\Gamma^\lambda_{\kappa\beta}+\partial_\kappa(\bar{g}^{\alpha\beta}\Gamma^\kappa_{\alpha\beta})-\partial_\alpha(\bar{g}^{\alpha\beta}\Gamma^\kappa_{\kappa\beta})]\mathrm{d}^dx$$
$$=\frac{1}{2\kappa}\int_B\bar{R}(\bar{g}^{\mu\nu},\partial_\kappa\bar{g}^{\mu\nu},\Gamma^\lambda_{\mu\nu})\ \mathrm{d}^dx+\frac{1}{2\kappa}\int_{\partial B}(\bar{g}^{\alpha\beta}\ \Gamma^\kappa_{\alpha\beta}-\bar{g}^{\kappa\beta}\ \Gamma^\alpha_{\alpha\beta})\ \mathrm{d}^{d-1}x_\kappa$$
$$=\ \frac{1}{2\kappa}\int_B\bar{R}\ \mathrm{d}^dx\ +\ \frac{1}{2\kappa}\int_{\partial B}K^\kappa\ \mathrm{d}^{d-1}x_\kappa \tag{D.63}$$

where $g_{\mu\nu}$ is the metric tensor, $R_{\mu\nu}(\Gamma^\lambda_{\mu\nu},\partial_\kappa\Gamma^\lambda_{\mu\nu})$ is the Ricci tensor as a function of the Christoffel symbol $\Gamma^\lambda_{\mu\nu}$, which is torsion free $\Gamma^\lambda_{\mu\nu}=\Gamma^\lambda_{(\mu\nu)}$, and its partial derivatives, and $\bar{g}^{\mu\nu}:=g^{\mu\nu}(-\det(g_{\mu\nu}))^{\frac{1}{2}}$ is the metric tensor density. $\kappa=8\pi G/c^4$ is the matter coupling constant. The integrand is the same as the original Lagrangian density: $\frac{1}{2\kappa}\ \bar{R}(\bar{g}^{\mu\nu},\partial_\kappa\bar{g}^{\mu\nu},\Gamma^\lambda_{\mu\nu}(\bar{g}^{\mu\nu},\partial_\kappa\bar{g}^{\mu\nu}))=\mathcal{L}_{EH}(g^{\mu\nu},\partial_\kappa g^{\mu\nu})$, up to the surface term.

The integrand in the last line has the appearance of a multiphase-space Lagrangian density:

$$\bar{R}(\bar{g}^{\mu\nu},\partial_\kappa\bar{g}^{\mu\nu},\Gamma^\lambda_{\mu\nu}):=-\partial_\kappa\bar{g}^{\alpha\beta}\ \Gamma^\kappa_{\alpha\beta}+\partial_\alpha\bar{g}^{\alpha\beta}\ \Gamma^\kappa_{\kappa\beta}-\bar{g}^{\alpha\beta}(\Gamma^\kappa_{\alpha\lambda}\Gamma^\lambda_{\kappa\beta}-\Gamma^\kappa_{\kappa\lambda}\Gamma^\lambda_{\alpha\beta}) \tag{D.64}$$

Following this observation, the last line is now treated as a multiphase-space first order action with Lagrangian $\bar{R}$ which is a function of field configuration degrees of freedom $\bar{g}^{\mu\nu}=\bar{g}^{(\mu\nu)}$, partial derivatives $\partial_\alpha\bar{g}^{\mu\nu}$, and (not canonical conjugate) multimomenta $-\Gamma^\alpha_{\mu\nu}=-\Gamma^\alpha_{(\mu\nu)}$:

$$S_{MP}[\bar{g}^{\mu\nu}(x),\Gamma^\lambda_{\mu\nu}(x)]=\int_B[\ \frac{1}{2\kappa}\ \bar{R}(\bar{g}^{\mu\nu},\partial_\kappa\bar{g}^{\mu\nu},\Gamma^\lambda_{\mu\nu})\ +\ \mathcal{L}_{fields}\ ]\ \mathrm{d}^dx\ +\ \frac{1}{2\kappa}\int_{\partial B}K^\kappa\ \mathrm{d}^{d-1}x_\kappa \tag{D.65}$$

Here a matter Lagrangian density $\mathcal{L}_{fields}$ has been added on at this point. The current $K^\kappa=(\bar{g}^{\alpha\beta}\ \Gamma^\kappa_{\alpha\beta}-\bar{g}^{\kappa\beta}\ \Gamma^\alpha_{\alpha\beta})$ is a surface term arising from the integration by parts, which will be ignored in the following discussion. This can be justified if the boundary is at infinity and spacetime becomes flat at infinity. For obtaining the Euler-Lagrange equations of motion, the fields $\bar{g}^{\alpha\beta}$ are fixed at the boundary, so we may ignore boundary terms in that case.

We calculate useful variations: $\delta g:=\delta\det(g_{\mu\nu})=gg^{\mu\nu}\delta g_{\mu\nu}=-gg_{\mu\nu}\delta g^{\mu\nu}$

$\delta\bar{g}^{\mu\nu}=(-g)^{\frac{1}{2}}(\delta^\mu_\alpha\delta^\nu_\beta-\frac{1}{2}g_{\alpha\beta}g^{\mu\nu})\delta g^{\alpha\beta}$

$\nabla^\nu f_\nu=(\partial^\nu+\Gamma_\alpha{}^{\alpha\nu})f_\nu$

If we assume that any Γ's in $\mathcal{L}_{fields}$ must be expressed in terms of the metric tensor g so that $\mathcal{L}_{fields}$ does not contain Γ's as independent degrees of freedom, the Euler-Lagrange equations for $\Gamma^\lambda_{\mu\nu}$ give the metric density compatibility condition for a Levi-Civita connection:

$$\partial_\kappa\bar{g}^{\alpha\beta}\ +2\Gamma^{(\alpha}_{\kappa\mu}\bar{g}^{\beta)\mu}-\Gamma^\mu_{\kappa\mu}\bar{g}^{\alpha\beta}=0, \tag{D.66}$$

while the Euler-Lagrange equations for $\bar{g}^{\mu\nu}$ are the Einstein field equations in the form involving the trace-reversed energy-momentum tensor:

$$\bar{R}_{\mu\nu} = \kappa(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\bar{T}_{\alpha\beta}) \tag{D.67}$$

or

$$R_{\mu\nu} = \kappa(T_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}T_{\alpha\beta}) = \kappa(T_{\mu\nu} - \frac{1}{2}g_{\mu\nu}g^{\alpha\beta}T_{\alpha\beta}) \tag{D.68}$$

where $T_{\alpha\beta} := \frac{1}{\sqrt{|\det(g_{\mu\nu}|)}}\frac{\delta\mathcal{L}_{fields}}{\delta g^{\alpha\beta}}$ is the energy-momentum tensor of $\mathcal{L}_{fields}$, and $\bar{T}_{\alpha\beta} := \frac{\delta\mathcal{L}_{fields}}{\delta g^{\alpha\beta}}$ is the energy-momentum tensor density of $\mathcal{L}_{fields}$.

D.4.1 Primary constraint

The Legendre transformation maps to multimomenta $p^{\alpha}_{\mu\nu}$ with

$$p^{\alpha}_{\mu\nu} \approx -\Gamma^{\alpha}_{\mu\nu} + \Gamma^{\kappa}_{\mu\kappa}\delta^{\alpha}_{\nu} = \Gamma^{\lambda}_{\mu\beta}\delta^{\beta\alpha}_{\lambda\nu} \tag{D.69}$$

which implies primary constraints:

$$p^{\alpha}_{\alpha\mu} \approx 0 \text{ and } (\delta^{\mu'\nu'}_{\mu\nu''})(\delta^{\alpha}_{\alpha'}\delta^{\nu''}_{\nu} \ - \ \frac{1}{d-1}\delta^{\nu''}_{\alpha'}\delta^{\alpha}_{\nu})\ p^{\alpha'}_{\mu'\nu'} = p^{\alpha}_{[\mu\nu]} - \frac{1}{d-1}p^{\kappa}_{[\mu|\kappa}\delta^{\alpha}_{|\nu]} \approx 0 \tag{D.70}$$

which arises from diffeomorphism invariance and the symmetry of the metric tensor density.

We can invert (D.69):

$$-\Gamma^{\alpha}_{\mu\nu} \approx p^{\alpha}_{\mu\nu} - \frac{1}{d-1}p^{\kappa}_{\mu\kappa}\delta^{\alpha}_{\nu} \tag{D.71}$$

Together with the second constraint, this implies the symmetry of the Christoffel symbol: $\Gamma^{\alpha}_{\mu\nu} = \Gamma^{\alpha}_{\nu\mu}$.

Substituting $\Gamma^{\alpha}_{\mu\nu} = -p^{\alpha}_{\mu\nu} + \delta^{\alpha}_{\nu}\Gamma^{\kappa}_{\kappa\mu}$ in $\bar{R}$ and K^{κ} , we can write the (up to a boundary term) Ricci scalar (D.64) as

$$\bar{R} = p^{\alpha}_{\mu\nu}\partial_{\alpha}\bar{g}^{\mu\nu} - \bar{g}^{\mu\nu}p^{\alpha}_{\mu\beta}p^{\beta}_{\alpha\nu} - p^{\beta}_{\beta(\nu}\partial_{\mu)}\bar{g}^{\mu\nu} \tag{D.72}$$

$$\bar{R} = p^{\alpha}_{\mu\nu}\partial_{\alpha}\bar{g}^{\mu\nu} - \mathcal{H} =: \widetilde{R}(\bar{g}^{\mu\nu}, \partial_{\alpha}\bar{g}^{\mu\nu}, p^{\alpha}_{\mu\nu}) \text{ and } K^{\kappa} = -\bar{g}^{\mu\nu}p^{\kappa}_{\mu\nu} \tag{D.73}$$

D.4.2 DDW Hamiltonian

The DDW Hamiltonian term $\mathcal{H}$ can be written as

$$\begin{aligned}\mathcal{H}(\bar{g}^{\mu\nu}, p^{\alpha}_{\mu\beta}) = \bar{g}^{\mu\nu}p^{\alpha}_{\mu\beta}p^{\beta}_{\alpha\nu} + p^{\beta}_{\beta(\nu}\partial_{\mu)}\bar{g}^{\mu\nu} = \bar{g}^{\mu\nu}p^{2}_{\mu\nu} \ + \ p^{\alpha}_{\alpha(\nu}\partial_{\beta)}\bar{g}^{\beta\nu} = \bar{g}^{\mu\nu}p^{2}_{\mu\nu} - \partial_{(\beta}p^{\alpha}_{|\alpha|\nu)}\bar{g}^{\beta\nu} \\ = \bar{g}^{\mu\nu}(p^{\alpha}_{\mu\beta}p^{\beta}_{\alpha\nu} - \partial_{\mu}p^{\alpha}_{\alpha\nu})\end{aligned} \tag{D.74}$$

employing integration by parts for the last term and where we define

$$p^{2}_{\mu\nu} \ := \ p^{\alpha}_{(\mu|\beta}p^{\beta}_{\alpha|\nu)} \ = \ \frac{1}{2}(\ \delta^{\mu'}_{\mu}\delta^{\nu'}_{\nu} + \delta^{\nu'}_{\mu}\delta^{\mu'}_{\nu})(\delta^{b}_{\alpha}\delta^{a}_{\beta}\ )\ p^{\alpha}_{\mu a}p^{\beta}_{b\nu'} \tag{D.75}$$

Integration by parts is performed inside the action integral for the second expression for $\mathcal{H}$, and the boundary term is ignored. This $\mathcal{H}(\bar{g}^{\mu\nu}, \partial_\alpha \bar{g}^{\mu\nu}, p^\alpha_{\mu\nu})$ is the DeDonderWeyl Hamiltonian for General Relativity, with the field variables being the elements of the metric tensor density $\bar{g}^{\mu\nu}$, the multimomenta being trace adjusted Christoffel symbols $p^\alpha_{\mu\nu}$, the momentum- stress of the spacetime metric density field, with the usual terms with derivatives when there are primary constraints. It should be noted that $p^\alpha_{\mu\nu}$, like Christoffel symbols, does not transform like a tensor or a tensor density, and contains physical information about the coordinate system together with the metric.

The DeDonder Weyl equations of motion are:

$$\begin{aligned} \partial_\alpha \bar{g}^{\mu\nu}(x) - \frac{\partial \mathcal{H}}{\partial p^\alpha_{\mu\nu}}(\bar{g}^{\kappa\lambda}(x), p^\beta_{\kappa\lambda}(x)) &\approx 0 \qquad \text{(DDW1)} \\ \partial_\alpha p^\alpha_{\mu\nu}(x) + \frac{\partial \mathcal{H}}{\partial \bar{g}_{\mu\nu}}(\bar{g}^{\kappa\lambda}(x), p^\beta_{\kappa\lambda}(x)) &\approx 0 \qquad \text{(DDW2)} \end{aligned} \tag{D.76}$$

which are, substituting for $\mathcal{H}$ above,

$$\begin{aligned} \partial_\alpha \bar{g}^{\mu\nu} \; - \; (\bar{g}^{\mu\kappa} p^\nu_{\alpha\kappa} + \bar{g}^{\kappa\nu} p^\mu_{\kappa\alpha}) - \delta^\mu_\alpha \partial_\beta \bar{g}^{\beta\nu} &\approx 0 \qquad \text{(DDW1)} \\ \partial_\alpha p^\alpha_{\mu\nu} \; + \; p^2_{\mu\nu} - \partial_{(\mu} p^\alpha_{|\alpha|\nu)} &\approx 0 \qquad \text{(DDW2)} \end{aligned} \tag{D.77}$$

The first equation (DDW1) is equivalent to the Levi-Civita compatibility condition between the metric and the Christoffel symbols, as can be seen by substituting for the multimomenta p using equation (D.69). Similarly the second equation (DDW2) is equivalent to the vacuum Einstein field equations. In the presence of matter, the zero on the right hand of DDW2 side is replaced by the trace adjusted energy momentum tensor density $\kappa(\bar{T}_{\mu\nu} - \frac{1}{2}\bar{g}_{\mu\nu}\bar{g}^{\alpha\beta}\bar{T}_{\alpha\beta})$ (assuming that the matter part of the DeDonder Weyl Hamiltonian does not contain p's and is the same as (D.68) once the substitution (D.69) is performed). These equations above (D.77) are the Euler-Lagrange equations for a stationary point of the following first order action:

$$\begin{aligned} S_{MP} &= \frac{1}{2\kappa}\int_{\Gamma J^1\mathcal{E}^*} (p^\alpha_{\mu\nu}\partial_\alpha \bar{g}^{\mu\nu} - \mathcal{H}) d^d x + \frac{1}{2\kappa}\int_{\partial\Gamma J^1\mathcal{E}^*} \bar{K}^\kappa \, \mathrm{d}^{d-1}x_\kappa \\ &= \frac{1}{2\kappa}\int_{\Gamma J^1\mathcal{E}^*} (p^\alpha_{\mu\nu}\partial_\alpha \bar{g}^{\mu\nu} - \bar{g}^{\mu\nu} p^\alpha_{\mu\beta} p^\beta_{\alpha\nu} - p^\beta_{\beta(\nu}\partial_{\mu)}\bar{g}^{\mu\nu}) \, d^d x + \frac{1}{2\kappa}\int_{\partial\Gamma J^1\mathcal{E}^*} (-\bar{g}^{\alpha\beta} \; p^\kappa_{\alpha\beta}) \, \mathrm{d}^{d-1}x_\kappa \\ &= \frac{1}{2\kappa}\int_{\Gamma J^1\mathcal{E}^*} ((\delta^{\lambda\alpha}_{\mu\beta} p^\beta_{\lambda\nu})\partial_\alpha \bar{g}^{\mu\nu} - \bar{g}^{\mu\nu} p^\alpha_{\mu\beta} p^\beta_{\alpha\nu}) - \partial_\kappa(\bar{g}^{\alpha\beta} \; p^\kappa_{\alpha\beta})) \, d^d x \end{aligned} \tag{D.78}$$

which has variation

$$\begin{aligned} \delta S_{MP} &= \delta\frac{1}{2\kappa}\int_{\Gamma J^1\mathcal{E}^*} (p^\alpha_{\mu\nu}\partial_\alpha \bar{g}^{\mu\nu} - \mathcal{H}) d^d x + \delta\frac{1}{2\kappa}\int_{\partial\Gamma J^1\mathcal{E}^*} \bar{K}^\kappa \, \mathrm{d}^{d-1}x_\kappa \\ &= \frac{1}{2\kappa}\int_{\Gamma J^1\mathcal{E}^*} [(\partial_\alpha \bar{g}^{\mu\nu} - \frac{\partial \mathcal{H}}{\partial p^\alpha_{\mu\nu}})\delta p^\alpha_{\mu\nu} - (\partial_\alpha p^\alpha_{\mu\nu} + \frac{\partial \mathcal{H}}{\partial \bar{g}_{\mu\nu}})\delta \bar{g}^{\mu\nu} + \frac{\partial \mathcal{H}}{\partial x^\mu}\delta x^\mu] d^d x \\ &\quad + \frac{1}{2\kappa}\int_{\partial\Gamma J^1\mathcal{E}^*} [-\bar{g}^{\mu\nu}\delta p^\alpha_{\mu\nu} \; + \; (p^\alpha_{\mu\nu}\partial_\alpha \bar{g}^{\mu\nu} - \mathcal{H})\delta x^\alpha \;] \, \mathrm{d}S_\alpha \end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{2\kappa}\int_{\Gamma J^1\mathcal{E}^*}(\partial_\alpha \bar{g}^{\mu\nu} - \ (\bar{g}^{\mu\kappa}p^{\nu}_{\alpha\kappa} + \bar{g}^{\kappa\nu}p^{\mu}_{\kappa\alpha}))\delta p^{\alpha}_{\mu\nu}\ d^dx\ - \\
&\quad \frac{1}{2\kappa}\int_{\Gamma J^1\mathcal{E}^*}(\partial_\alpha p^{\alpha}_{\mu\nu} - \ p^{\alpha}_{(\mu|\beta}p^{\beta}_{\alpha|\nu)}\ )\delta\bar{g}^{\mu\nu}\ d^dx \\
&+ \frac{1}{2\kappa}\int_{\partial\Gamma J^1\mathcal{E}^*}[\,-\bar{g}^{\mu\nu}\delta p^{\alpha}_{\mu\nu}\ +\ (p^{\alpha}_{\mu\nu}\partial_\alpha\bar{g}^{\mu\nu} - \bar{g}^{\mu\nu}p^{\alpha}_{\mu\beta}p^{\beta}_{\alpha\nu})\delta x^{\alpha}\ ]\mathrm{d}S_\alpha
\end{aligned} \tag{D.79}
$$

together with the primary constraints, $p^{\alpha}_{\alpha\mu} = 0$.

A vector field $X = f^{\mu}(x)\frac{\partial}{\partial x^{\mu}}$ is an infinitesimal diffeomorphism of the spacetime manifold which may be viewed as infinitesimal coordinate changes, $x^{\mu} + \delta_f x^{\mu} = x^{\mu} + f^{\mu}$, which preserve the metric if the changes in the components of the metric is given by the Lie derivative of the metric, with respect to the vector field X:

$$
\delta_f g_{\mu\nu} = \mathcal{L}_X g_{\mu\nu} = 2\nabla_{(\mu}f_{\nu)}(x) = f^{\kappa}g_{\mu\nu,\kappa} + f^{\kappa}{}_{,\mu}g_{\nu\kappa} + f^{\kappa}{}_{,\nu}g_{\mu\kappa} \tag{D.80}
$$

and the variation of $g^{\mu\nu}$ is

$$
\begin{aligned}
\delta_f g^{\mu\nu} = \mathcal{L}_X g^{\mu\nu} &= -2\nabla_\kappa f^{(\nu}g^{\mu)\kappa}(x) = (-f^{\kappa}g^{\mu\nu}{}_{,\kappa} - f^{\mu}{}_{,\kappa}g^{\nu\kappa} - f^{\nu}{}_{,\kappa}g^{\mu\kappa}) \\
&= 2(-f^{\kappa}g^{\lambda(\nu}\Gamma^{\mu)}_{\kappa\lambda} - f^{(\mu}{}_{,\kappa}g^{\nu)\kappa})
\end{aligned} \tag{D.81}
$$

where $f_\nu = f^{\mu}g_{\mu\nu}$.

The equivalent variation in $\bar{g}^{\mu\nu}$ is

$$
\begin{aligned}
\delta_f\bar{g}^{\kappa\lambda} &= -(2\nabla_\kappa f^{(\nu}\bar{g}^{\mu)\kappa} + \nabla_\kappa f^{\kappa}\bar{g}^{\mu\nu})(x) \\
&= -2(f^{\kappa}\bar{g}^{\lambda(\nu}\Gamma^{\mu)}_{\kappa\lambda} + f^{(\mu}{}_{,\kappa}\bar{g}^{\nu)\kappa}) - (f^{\kappa}\bar{g}^{\mu\nu}\Gamma^{\lambda}_{\kappa\lambda} + f^{\kappa}{}_{,\kappa}\bar{g}^{\mu\nu}) \\
&= -(f^{\nu}\bar{g}^{\kappa\lambda}{}_{,\nu} + 2f^{(\kappa}{}_{,\nu}\bar{g}^{\lambda)\nu} + f^{\nu}{}_{,\nu}\bar{g}^{\kappa\lambda} - \frac{1}{d+2}f^{\kappa}\bar{g}^{\kappa\lambda}\bar{g}^{\alpha\beta}{}_{,\nu}\bar{g}_{\alpha\beta})
\end{aligned} \tag{D.82}
$$

The extra terms are due to the presence of the volume factor $g^{\frac{1}{2}}$ in $\bar{g}^{\mu\nu} = g^{\frac{1}{2}}g^{\mu\nu}$. Replacing $\Gamma^{\alpha}_{\mu\nu}$ by multimomenta using (D.71), $-\Gamma^{\alpha}_{\mu\nu} \approx p^{\alpha}_{\mu\nu} - \frac{1}{d-1}p^{\kappa}_{\mu\kappa}\delta^{\alpha}_{\nu}$, gives

$$
\delta_f\bar{g}^{\kappa\lambda}\ \approx\ 2\ (f^{\kappa}\bar{g}^{\lambda(\nu}p^{\mu)}_{\kappa\lambda} - f^{(\mu}{}_{,\kappa}\bar{g}^{\nu)\kappa}) + (-\frac{3}{d-1}f^{\kappa}\bar{g}^{\mu\nu}p^{\lambda}_{\kappa\lambda} - f^{\kappa}{}_{,\kappa}\bar{g}^{\mu\nu}) \tag{D.83}
$$

The constraints $P^{\gamma}_{\nu} := \bar{g}^{\gamma\mu}\partial_{(\mu}p^{\alpha}_{|\alpha|\nu)} = 0$ generates the gauge transformations , $\delta_f\bar{g}^{\kappa\lambda} = 2\partial_\mu f^{(\lambda}\bar{g}^{\kappa)\mu} + f^{\nu}\partial_\nu\bar{g}^{\lambda\kappa} - \partial_\nu f^{\nu}\bar{g}^{\lambda\kappa} = 2\partial^{(\lambda}f^{\kappa)} + f^{\nu}\partial_\nu\bar{g}^{\lambda\kappa} - \partial_\nu f^{\nu}\bar{g}^{\lambda\kappa}$, under which the original action (D.63) and the first order action (D.78) are invariant (because these gauge transformations are diffeomorphisms which are active coordinate transformations) , via the multi-bracket:

$$
\begin{aligned}
-\delta_f\bar{g}^{\kappa\lambda} = \{\bar{g}^{\gamma\mu}f^{\nu}\partial_{(\mu}p^{\beta}_{|\beta|\nu)}, \bar{g}^{\kappa\lambda}\}_\gamma = \mathrm{d}_V(\bar{g}^{\gamma\mu}f^{\nu}\partial_{(\mu}p^{\beta}_{|\beta|\nu)})\llcorner\Pi_\gamma\lrcorner\mathrm{d}_V(\bar{g}^{\kappa\lambda}) = \\
\frac{1}{2}\bar{g}^{\gamma\mu}(\bar{f}^{\nu}\partial_\mu p^{\beta}_{\beta\nu} + f^{\nu}\partial_\nu p^{\beta}_{\beta\mu})\cdot(\frac{\overleftarrow{\partial}}{\partial\bar{g}^{\rho\sigma}} \wedge \frac{\overrightarrow{\partial}}{\partial p^{\gamma}_{\rho\sigma}})\cdot(\bar{g}^{\kappa\lambda})
\end{aligned}
$$

$$= f^\nu(\partial_\mu \bar{g}^{\gamma\mu} p^\beta_{\beta\nu} + \partial_\nu \bar{g}^{\gamma\mu} p^\beta_{\beta\mu}) + \frac{1}{2}\bar{g}^{\gamma\mu}(\partial_\mu f^\nu p^\beta_{\beta\nu} + \partial_\nu f^\nu p^\beta_{\beta\mu}) \cdot (\frac{\overleftarrow{\partial}}{\partial \bar{g}^{\rho\sigma}} \wedge \frac{\overrightarrow{\partial}}{\partial p^\gamma_{\rho\sigma}}) \cdot (\bar{g}^{\kappa\lambda}) =$$

$$-\frac{1}{2}(f^\nu \partial_\mu \bar{g}^{\gamma\mu} + \bar{g}^{\gamma\mu}\partial_\mu f^\nu)\left(\frac{\partial p^\beta_{\beta\nu}}{\partial p^\gamma_{\rho\sigma}}\right)\left(\frac{\partial \bar{g}^{\kappa\lambda}}{\partial \bar{g}^{\rho\sigma}}\right) - \frac{1}{2}(f^\nu \partial_\nu \bar{g}^{\gamma\mu} + \bar{g}^{\gamma\mu}\partial_\nu f^\nu)\left(\frac{\partial p^\beta_{\beta\mu}}{\partial p^\gamma_{\rho\sigma}}\right)\left(\frac{\partial \bar{g}^{\kappa\lambda}}{\partial \bar{g}^{\rho\sigma}}\right)$$

$$= -\frac{1}{2}\ ((f^\nu \partial_\mu \bar{g}^{\gamma\mu} + \bar{g}^{\gamma\mu}\partial_\mu f^\nu)\delta^\beta_\gamma \delta^\rho_\beta \delta^\sigma_\nu + (f^\nu \partial_\nu \bar{g}^{\gamma\mu} + \bar{g}^{\gamma\mu}\partial_\nu f^\nu)\delta^\beta_\gamma \delta^\rho_\beta \delta^\sigma_\mu)\ (\delta^{(\kappa}_\rho \delta^{\lambda)}_\sigma)$$

$$= -\frac{1}{2}\ (\ \ \bar{g}^{\gamma\mu}(\partial_\mu f^{(\lambda}\delta^{\kappa)}_\gamma + \partial_\nu f^\nu \delta^{(\lambda}_\mu \delta^{\kappa)}_\gamma) + \ f^\nu \partial_\nu \bar{g}^{\kappa\lambda} + f^{(\kappa}\partial_\mu \bar{g}^{\lambda)\mu}\ \ )$$

$$= -\frac{1}{2}\ (\ \ \partial_\mu f^{(\lambda}\bar{g}^{\kappa)\mu} + \partial_\nu f^\nu \bar{g}^{\lambda\kappa} + \ f^\nu \partial_\nu \bar{g}^{\kappa\lambda} + f^{(\kappa}\partial_\mu \bar{g}^{\lambda)\mu}\ \ ) \tag{D.84}$$

The variation of the multimomenta is:

$$-\delta_f p^\lambda_{\rho\sigma} = \{\bar{g}^{\gamma\mu} f^\nu \partial_{(\mu} p^\beta_{|\beta|\nu)}, p^\lambda_{\rho\sigma}\}_\gamma = \delta^\lambda_\rho f^\nu \partial_{(\sigma} p^\beta_{|\beta|\nu)}\ \ = \delta^\lambda_\rho \bar{g}_{\sigma\gamma} P^\gamma_\nu \tag{D.85}$$

which is zero on the constraint surface $P^\gamma_\nu = 0$.

By employing integration by parts in the multiphase-space action the constraints as $P^\gamma(f) := f^\nu \bar{g}^{\gamma\mu}\partial_{(\mu} p^\alpha_{|\alpha|\nu)} = 0$ can be replaced by

$$P^{'\gamma} := -\frac{1}{2}[f^\nu(\partial_\mu \bar{g}^{\gamma\mu} p^\beta_{\beta\nu} + \partial_\nu \bar{g}^{\gamma\mu} p^\beta_{\beta\mu}) + \bar{g}^{\gamma\mu}(\partial_\mu f^\nu p^\beta_{\beta\nu} + \partial_\nu f^\nu p^\beta_{\beta\mu})] = 0 \tag{D.86}$$

and the partial derivatives $\partial_\nu \bar{g}^{\gamma\mu}$ can be replaced by linear functions of the multimomenta by using the equations of motion, so that we have constraints defined on multiphase space:

$$P^{''\gamma}(f) :=$$

$$-\frac{1}{2}[f^\nu(\bar{g}^{\mu\kappa} p^\gamma_{\kappa\mu} p^\beta_{\beta\nu} + \bar{g}^{\kappa\gamma} p^\mu_{\kappa\mu} p^\beta_{\beta\nu} + \bar{g}^{\kappa\gamma} p^\mu_{\nu\kappa} p^\beta_{\beta\mu} + \bar{g}^{\kappa\mu} p^\gamma_{\kappa\nu} p^\beta_{\beta\mu}) + \bar{g}^{\gamma\mu}(\partial_\mu f^\nu p^\beta_{\beta\nu} + \partial_\nu f^\nu p^\beta_{\beta\mu})]$$

$$=\ \ -2\ f^\nu \bar{g}^{\kappa(\mu} p^{\gamma)}_{\kappa(\nu} p^\beta_{|\beta|\mu)}\ \ -\ \ \bar{g}^{\gamma\mu}\partial_{(\mu} f^\nu p^\beta_{|\beta|\nu)}\ \ =\ \ 0 \tag{D.87}$$

which is quadratic in the multimomenta, like the example preceeding this one of of the bosonic string.

Appendix E

Multivector picture

Because the solutions of the field equations do not foliate multiphase space in the way that the phase space Hamiltonian generates a vector field on phase space, which can be integrated to form a unique foliation, the geometry of the solutions in multiphase space is more complicated. This section examines how solutions are characterized in terms of multivector fields describing the tangent to the section representing the solution in the multiphase space bundle.

Appendix H describes multiphase space Hamilton-Jacobi theory which is also relevant to classifying solutions. This is preceeded by appendix G which deals with Hamilton-Jacobi theory in phase space.

The question of the equations of motion expressed in terms of multivector fields and of the integrability of the latter is described by A. Echeverria-Enriquez, M.C. Munoz-Lecanda, N. Roman-Roy [2] and in 'Geometry of Hamiltonean n-vectorfields in Multisymplectic Field Theory' by Cornelius Paufler and Hartmann Romer [41].

Here we examine multivectors as a multisymplectic field analogue to a velocity vector in phase space and which may be useful in interpreting multisymplectic formalism geometrically.

An important property of the Hamiltonian formulation of mechanics is that the solutions of the equations of motion (for a time independant Hamiltonian) foliate the time-extended phase space into distinct trajectories: no two trajectories have any point in common. This arises from the properties of first order ordinary differential equations (Hamilton's equations of motion), which contrasts with the situation in multiphase space where the equations of motion are partial differential equations over a d-dimensional spacetime. In particular, in Hamiltonian mechanics, the possible evolutions of the system can be described by a foliation of the phase-space fiber bundle over time: $\mathfrak{T}^*Q \times \mathfrak{R}$ into classical trajectories, by integrating the vector field $X_H + \frac{\partial}{\partial t}$ specified by Hamilton's equations (2.1), equivalent to the symplectic equation of motion (2.2), with a given Hamiltonian function $H(q, p, t)$. The space of trajectories, in the regular case, can be represented by taking a time $t = t_0$ slice of time-extended phase space which intersects each trajectory at a single point $(q(t_0), p(t_0))$. Thus the space of trajectories, in the regular case, is a 1-1 map to phase space. In the Lagrangian system with a given Lagrangian function, a specific trajectory is selected by specifying the configuration at the endpoints of a system evolving over the time interval $\Delta T = [t_i, t_f]$: $\{q(t_i), q(t_f)\}$, and, in the regular case, the space of trajectories, in that time interval, has a 1-1 map to $Q^{\partial \Delta T} := Q \times Q$.

Looking at fields, it is of interest whether one can have foliations of multiphase space where each leaf is a solution to the DDW field equations. One approach is dealt with in appendix H using Hamilton-Jacobi theory. Another is approach is that in this appendix.

In classical field theory, a particular field solution $\phi(x)$ is a section $\Gamma_\phi^{\mathcal{E}} \in \Gamma(B, \mathcal{E})$ of the configuration bundle $\mathcal{E} \longrightarrow B:$ a spacetime-horizontal d-dimensional submanifold of the total space $\mathcal{E}$. This canonically prolongs to a section $\Gamma_\phi^{J^1\mathcal{E}} \in \Gamma(B, J^1\mathcal{E})$ of the first jet bundle $J^1\mathcal{E}$, and, via the Legendre transformation, to a section $\Gamma_\phi^{\mathcal{M}} \in \Gamma(B, \mathcal{M})$ of the multiphase space $\mathcal{M}$. Thus the structures analogous to trajectories in phase space are (spacetime-horizontal) d-dimensional submanifolds of the multiphase space. However the set of all solutions to the field equations is not given by a foliation of the multiphase space (for instance, different solutions may share the same values of the fields and multimomenta at a specific point spacetime). However it is of interest to have families of solutions, if not all the solutions, which foliate configuration or multiphase space.

In a field theory with a specified Lagrangian density, the space of trajectories (spacetime field configurations which satisfy the action principle modulo the gauge symmetries) in a spacetime region B' is 1-1 with the space of field $U^{\partial B'}$ values defines on the boundary $u : \partial B' \longmapsto U$. How these are obtained is shown in this section by employing multivectors. In appendix H, it will be achieved via a generalization of Hamilton-Jacobi theory, by employing a map from the configuration space to the multiphase space, $\mathbb{T} : \mathcal{E} \longrightarrow \mathcal{M} \cong \Lambda_1^n \mathcal{E}$, which pulls back the multivector field on $\mathcal{M}$ to an integrable multivector field on $\mathcal{E}$ which foliates $\mathcal{E}$ locally. For this to be the case $\mathbb{T}$ has to satisfy the generalized Hamilton-Jacobi equations.

We are interested in multivectors which define the tangent subspace of the embedding of a section over spacetime (which represents a field configuration) in the bundles involved in multisymplectic dynamics. But first we consider the general case of a submanifold Γ embedded in a general manifold $\mathcal{M}$. The d-dimensional tangent subspace $D^d_{\Gamma m} \subset \mathfrak{T}_m\mathcal{M}$ at each point m of an d-dimensional submanifold Γ embedded in a manifold $\mathcal{M}$ defines on Γ a decomposable (also called simple) d-multivector $Y \in \Lambda^d \mathfrak{T} M$ unique up to multiplication by a function $f(m)$. (A set of subspaces $D^d_{\Gamma m}$ of the tangent space $\mathfrak{T}_m$ over all points m in a submanifold Γ is called a distribution. A decomposable (also called simple) d-multivector $Y \in \Lambda^d \mathfrak{T} M$ is a multivector which can be written as an anti-symmetric product of d vectors: $Y = Y_1 \wedge ... \wedge Y_d$). This multivector field on the section Γ is the generalization of the vector field $X_H + \frac{\partial}{\partial t}$, mentioned above, along a particular path in a symplectic manifold. The multivector fields will satisfy certain conditions: they have to be locally decomposable to an anti-symmetric product $Y = f(m) Y_1 \wedge ... \wedge Y_d$ of vectors , $Y_\mu \in \mathfrak{T}\Gamma$ to specify an d-dimensional tangent space of a d-dimensional subspace of $\mathcal{M}$. In addition, the tangent subspaces should integrate to give smooth submanifolds, so the vector fields Y_μ should be involutive (to be integrable to generate a smooth function of spacetime): $[Y_\mu, Y_\nu] = g^\tau_{\mu\nu}(m) Y_\tau$, by Frobenius' theorem. It is possible to specify $f(m)$ on $\mathcal{M}$ so that the Y_μ can be chosen locally, to make their Lie brackets with each other be zero. Thereby these vectors integrate to a local coordinate system on the submanifold Γ (called a dynamical multivector field by [2]). On a section, these local coordinate functions are mapped by the bundle projection (which is bijective on a section) to a local coordinate system on the base space.

The instances of interest here are $Y_\mu = \frac{\partial \bar{\phi}}{\partial x^\mu}$, where $\bar{\phi}$ is $\bar{\phi}(x)$ embedded in $\mathcal{E}$ or a canonical prolongation embedded in the first jet bundle $J^1\mathcal{E}$ or the multiphase space $\mathcal{M}$.

H. Romer and C. Paufler [41] have shown by construction that, given any smooth function (0-form) H, on the multiphase space $\mathcal{M}$ (with locally adapted coordinates x^μ, u^i, p^μ_i, p), of the form

$$H(x^\mu, u^i, p^\mu_i, p) = -\mathcal{H}(x^\mu, u^i, p^\mu_i) - p \tag{E.1}$$

there are decomposable d-multivector fields Y_H on $\mathcal{M}$, which can be viewed as distributions on $\mathcal{M}$, which are multi-hamiltonian for H:

$$Y_H \lrcorner \Omega = \mathrm{d}H, \quad \text{where} \quad Y_H = Y_1 \wedge ... \wedge Y_d \quad \text{and} \quad \Omega = -(\mathrm{d}p^\mu_i \wedge \mathrm{d}u^i \wedge \mathrm{d}x_\mu + \mathrm{d}p \wedge \mathrm{d}^d x) \tag{E.2}$$

Note that for a DDW Hamiltonian $\mathcal{H}(x^\mu, u^i, p^\mu_i)$, $H(x^\mu, u^i, p^\mu_i, p) = 0$ on points (x^μ, u^i, p^μ_i, p) in the image $\mathcal{P}_{\mathbb{L}}$ of the Legendre transformation (3.37), where $p = -\mathcal{H}(x^\mu, u^i, p^\mu_i(x^\mu, u^i, u^i_\mu))$, by definition.

The d-multivector Y_H is hamiltonian, $Y_H \lrcorner \Omega = \mathrm{d}H$, for the function $H(x^\mu, u^i, p^\mu_i, p) = -\mathcal{H}(x^\mu, u^i, p^\mu_i) - p$ if

$$Y_H = Y_1 \wedge ... \wedge Y_d \tag{E.3}$$

and the components Y_μ satisfy (E.6).

This can be shown by explicitly calculating both sides of the equation (E.2) and identifying coefficients of basis vectors: Expanding the vectors of the decomposition in terms of coordinate basis vectors,

$$Y_\mu = Y_\mu^\nu \bar{e}_\nu + Y_\mu^i \bar{e}_i + Y_{\mu i}{}^\nu \bar{e}_\nu^i + Y_\mu^{(p)} \bar{e}_p \tag{E.4}$$

(E.2) gives for the coefficient of the spacetime multivector,

$$\begin{aligned} {}^d\partial &:= \frac{\partial}{\partial x^{d-1}} \wedge \frac{\partial}{\partial x^{d-2}} \wedge \ldots \wedge \frac{\partial}{\partial x^2} \wedge \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^0} \\ &= (-1)^{d(d-1)/2} \frac{\partial}{\partial x^0} \wedge \frac{\partial}{\partial x^1} \wedge \frac{\partial}{\partial x^2} \wedge \ldots \wedge \frac{\partial}{\partial x^{d-2}} \wedge \frac{\partial}{\partial x^{d-1}} \end{aligned} \tag{E.5}$$

the following identity, where the determinant of Y_H refers to the mapping from the tangent space of spacetime to itself induced by the bundle projection of Y_H to spacetime.

$$Y_1^{\mu_1} \ldots Y_d^{\mu_d} \epsilon_{\mu_1 \ldots \mu_d} =: \det(Y_H) = (-1)^{d+1} \frac{\partial H}{\partial p} =: (-1)^{d+1} \partial_p H \tag{E.6}$$

This shows that for the Y_μ's to be linearly independent, $\partial_p H \neq 0$, at any point requires $\partial_p H$ to be a monotonic function of p. If we make the natural choice of the coefficients of the spacetime part of the decomposition vectors to be 1, $Y_\mu^\nu \bar{e}_\nu = -\delta_\mu^\nu \bar{e}_\nu$, then $\partial_p H = -1$, which explains the particular definition of the p dependence of H.

The complete vector decomposition which satisfies the multisymplectic equation (E.2) (and which bundle projects to (minus the) coordinate basis vectors on spacetime) is

$$Y_\mu = -\bar{e}_\mu - (\partial_\mu^i H)\bar{e}_i - (\frac{1}{d}\delta_\mu^\nu \partial_i H - A_{\mu i}^\nu)\bar{e}_\nu^i - ((\partial_\mu^i H)(\partial_i H) + (\partial_\nu^i H)(\frac{1}{d}\delta_\mu^\nu \partial_i H - A_{\mu i}^\nu) + \partial_\mu H)\bar{e}_p \tag{E.7}$$

where $A_{\mu i}^\nu(x, u^i, p_i^\mu, p)$ are any functions which satisfy $A_{\nu,i}^\nu = 0$, $\bar{e}_\mu, \bar{e}_i, \bar{e}_\mu^i, \bar{e}_p$ are the coordinate basis vectors for the local coordinates x, u^i, p_i^μ, p in multiphase space. $\partial_\mu, \partial_i, \partial_\mu^i, \partial_p$ are the partial derivatives along coordinate lines in the directions $\bar{e}_\mu, \bar{e}_i, \bar{e}_\mu^i, \bar{e}_p$. Also, the determinant of the bundle projection of Y_H to the spacetime base space is

$$\det(Y_H) = Y_1^{\mu_1} \ldots Y_d^{\mu_d} \epsilon_{\mu_1 \ldots \mu_d} = (-1)^{d+1} \partial_p H = (-1)^d \tag{E.8}$$

where $Y_\nu^\mu = -\delta_\nu^\mu$ are the spacetime components of the vector Y_ν, i.e. the coefficients of $\bar{e}_\mu$ in (E.7). Thus the d vectors Y_k are linearly independent and so span a rank d distribution on $\mathcal{M}$. The bundle projection of the vector Y_k to spacetime B is $Y_k^B = -\bar{e}_\mu$ and so the d vector fields $Y_k^B(x)$, which are the negative local coordinate basis vector fields on B, are linearly independent. Note that $0 = (Y_\mu \wedge Y_H) \lrcorner \Omega = Y_\mu \lrcorner dH$ so the vectors Y_μ lie in the kernel of dH, i.e. the distribution Y_H lies inside the tangent to the constant hypersurfaces of the function H on $\mathcal{M}$.

From (E.2), the determinant of a decomposable Hamiltonian multivector field, canonically projected to the (space-time) base manifold, is equal to $\frac{\partial H}{\partial p}$ so the form of (E.1) corresponds to a choice of the determinant to be equal to $(-1)^d$, corresponding to an energy scale p adapted to the volume on space-time.

Because of the freedom (up to $A^{\nu}_{\nu,i} = 0$) to choose the functions $A^{\nu}_{\mu i}(x, u^i, p^{\mu}_i, p)$ in (E.7), the multisymplectic equation of motion (E.2), $X_H \lrcorner \Omega = \mathrm{d}H$, does not determine unique decomposable multivector fields, but rather a large family of Hamiltonian multivector fields. For the physical fields we are considering we are seeking multivector fields that are integrable distributions. A field configuration, (section of $\mathcal{M}$ as a bundle over B) which satisfy the DDW equations of motion for $\mathcal{H}$, as a submanifold of $\mathcal{M}$ has tangent spaces which can be expressed by decomposable multivectors like Y_H above. Therefore these multivectors satisfy the multisymplectic equation of motion (E.2), $X_H \lrcorner \Omega = \mathrm{d}H$, defined on the submanifold. Conversely a distribution Y_H which satisfies $X_H \lrcorner \Omega = \mathrm{d}H$ and which is integrable on some region of $\mathcal{M}$ defines a foliation into local sections which individually satisfy the DDW equations.

Nor will there be a unique integrable distribution because, for example, there can be many different field trajectories which intersect a given point $(x^{\mu}, \phi^i, p^{\mu}_i, -\mathcal{H}(x^{\mu}, \phi^i, p^{\mu}_i))$ in multi-phase space $\mathcal{M}$. This is unlike the situation in Hamiltonian mechanics, where there is a single trajectory passing through any given point (t, q^i, p_i) in time extended phase space. Thus unlike time extended phase space, there is not a single foliation of trajectories, and foliations may not be possible anyway. This means that there will not be a unique choice of $A^{\nu}_{\nu,i}$ which make for integrable distributions.

A distribution Y_H may be viewed as an Ehresmann connection E on the bundle $\pi^{\mathcal{M},B} : \mathcal{M} \longrightarrow B$, because it defines a unique horizontal lift of every vector $\frac{\partial}{\partial x^{\mu}}$ on the base space defined by (E.7). To emphasize this we write $D_{\mu} := Y_{\mu}$, the vector fields can be viewed as the covariant derivative for the connection. We are interested in integrability of the distribution, which in terms of the covariant dervative, corresponds to the curvature $R_{\mu\nu} := [D_{\mu}, D_{\nu}]$ being zero, i.e. flat connections.

The equation $[Y_{\mu}, Y_{\nu}] = 0$, for given $\mathcal{H}$ are partial differential equations for the functions $A^{\nu}_{\mu,i}$, in addition to the condition $A^{\nu}_{\nu,i} = 0$. The solutions will be integrable distributions, although the integral surfaces will not form foliations. That means that the solutions may only exist on submanifolds of $\mathcal{M}$.

To pursue further the geometry of solutions of the DDW equations, appendix H presents multisymplectic Hamilton-Jacobi theory.

Appendix F

Canonical transformations

Coordinate changes on phase space which preserve the form of Hamiltons equations are called canonical transformations and are explained in this appendix. The multiphase-space generalization is in a later appendix.

In practice, dynamical systems are often specified in a particular coordinate system on phase space and it may be useful to change to a different coordinate system. This may help, for instance, solve the Hamilton's equations of motion, simplify the equations of motion, or explicitly embody a symmetry. A frequent aim is to eliminate some phase-space variables from the Hamiltonian, at which point the canonical duals will be constants of motion parametrizing solutions. Hamilton-Jacobi theory (below) can be thought of as involving a change to coordinates on time extended phase space which are constant on trajectories.

Canonical transformations $Q^i = Q^i(q,p,t), P_i = P_i(q,p,t)$, (where q stands for the full set of coordinates $q^i, i = 1\ldots, N$, and similarly for p, Q, P , here and in the following) are coordinate changes on the fibers on time-extended phase space $\mathfrak{T}^*Q' := \mathfrak{T}^*Q \times \mathbb{R}$ which preserve the form of Hamilton's equations. They are a particular type of symplectomorphism when viewed as coordinate change on a symplectic manifold. The special case of changing only the configuration coordinates $Q^i = Q^i(q,t)$ is called a *restricted* canonical transformation. Hamilton-Jacobi theory in the section following can be viewed as stemming from a particular type of canonical transformation.

We are seeking a transformation of coordinates in phase space or time-extended phase space from the original coordinates, (q,p,t), to new ones, (Q,P,t), while preserving the form of

Hamilton's equations:

$$\dot{Q}^i \;\approx\; \frac{\partial H}{\partial P_i}(\bar{Q},\bar{P},t) \quad , \quad \dot{P}_i \;\approx\; -\frac{\partial H}{\partial Q^i}(\bar{Q},\bar{P},t), \tag{F.1}$$

where the $\bar{Q} := (Q^i)_{i=1...N}$ are N configuration dynamical variables which are local coordinates of the configuration space Q , and $(P_i)_{i=1...N}$ are the momentum dynamical variables canonically dual to the Q^i. $(Q^i, P_i, t)_{i=1...N}$ are local adapted coordinates on the classical time-extended phase space of the system, $\mathfrak{T}^*Q'$, and are transformed coordinates $Q^i = Q^i(q,p,t), P_i = P_i(q,p,t)$. $H = H(Q,P,t)$ is a function on $\mathfrak{T}^*Q'$, which is the Hamiltonian function in the new coordinate system. $(\dot{Q}^i, \dot{P}_i, \dot{t} = 1)$ are the time rate of change of the dynamical variables. $h = h(q,p,t) = H(Q(q,p,t), P(q,p,t), t)$ is a function on $\mathfrak{T}^*Q'$, and is the Hamiltonian function in the original coordinates.

For notational simplicity we restrict to the case $N = 1$. We will also employ a compact notation- where $Z = Z(z,t)$ is shorthand for either $Q = Q(q,p,t) or P = P(q,p,t)$, and W is canonically dual to Z etc, so, in an equation, if $z = p$ then $w = q$ and vice versa.

Starting with coordinate free notation we express the equation of motion for $\dot{Z} = \dot{Q}$ and $\dot{P}$ using Poisson brackets expressed in first the old, and secondly the new coordinate systems,

$$\dot{Z} \approx \{Z,H\} = \frac{\partial Z}{\partial q}\frac{\partial H}{\partial p} - \frac{\partial Z}{\partial p}\frac{\partial H}{\partial q} + \frac{\partial Z}{\partial t}\ (q,p,t) \qquad \text{(old)} \tag{F.2}$$

$$\begin{aligned}\dot{Z} \approx \{Z,H\} &= \pm\frac{\partial Z}{\partial Z}\frac{\partial H}{\partial W} + \frac{\partial Z}{\partial t}\ (Q,P,t) \qquad \text{(new)} \\ &= \pm\Big(\frac{\partial H}{\partial q}\frac{\partial q}{\partial W} + \frac{\partial H}{\partial p}\frac{\partial p}{\partial W}\Big) + \frac{\partial Z}{\partial t}\ (Q,P,t)\end{aligned} \tag{F.3}$$

where (q,p,t) and (Q,P,t) are the same point in time-extended phase space in the different coordinate systems. Note: that in the second line $\frac{\partial Z}{\partial W} = 0$. Identifying the coefficients of $\frac{\partial H}{\partial q}$ and $\frac{\partial H}{\partial p}$ we obtain

$$\begin{aligned}\frac{\partial Q}{\partial q}\Big|_{p,t}(q,p) &= \frac{\partial p}{\partial P}\Big|_{Q,t}(Q,P) & -\frac{\partial Q}{\partial p}\Big|_{q,t}(q,p) &= \frac{\partial q}{\partial P}\Big|_{Q,t}(Q,P) \\ -\frac{\partial P}{\partial q}\Big|_{p,t}(q,p) &= \frac{\partial p}{\partial Q}\Big|_{P,t}(Q,P) & \frac{\partial P}{\partial p}\Big|_{q,t}(q,p) &= \frac{\partial q}{\partial Q}\Big|_{P,t}(Q,P)\end{aligned} \tag{F.4}$$

Both sides of these equations refer to the same point in time-extended phase space, but employing different coordinate systems.

These are just the conditions on coordinate transformations to be a symplectomorphism: $AJA^T = J$ where $A^r{}_s = \frac{\partial Z^r}{\partial z^s}$, the $2N \times 2N$ Jacobian matrix of the transformation from z to Z, and J is symplectic form,

$$J = \begin{pmatrix} 0 & I_N \\ -I_N & 0 \end{pmatrix} \tag{F.5}$$

where I_N is the $N \times N$ identity matrix.

The conditions (F.4) can be reproduced by calculating $A = JA^{T^{-1}}J^{-1}$:

$$A = \begin{pmatrix} \frac{\partial Q}{\partial q}\Big|_{p,t}(q,p) & \frac{\partial Q}{\partial p}\Big|_{q,t}(q,p) \\ \frac{\partial P}{\partial q}\Big|_{p,t}(q,p) & \frac{\partial P}{\partial p}\Big|_{q,t}(q,p) \end{pmatrix} = JA^{T^{-1}}J^{-1} = \begin{pmatrix} \frac{\partial p}{\partial P}\Big|_{Q,t}(Q,P) & -\frac{\partial q}{\partial P}\Big|_{Q,t}(Q,P) \\ -\frac{\partial p}{\partial Q}\Big|_{P,t}(Q,P) & \frac{\partial q}{\partial Q}\Big|_{P,t}(Q,P) \end{pmatrix} \tag{F.6}$$

F.1 Method of generating functions

We can obtain canonical transformations by expressing the action using the phase-space Lagrangian formalism in both the original and the new coordinate system on phase space.

We want new coordinates $Q^i = Q^i(q,p,t), P_i = P_i(q,p,t), T = T(t)$ on the bundle of the phase space over time $\mathfrak{T}^*Q \times \mathbb{R}$. We also want the Hamiltonian in the new coordinates, $H = H(Q,P,T)$, which preserves the phase-space action along any curve C in phase space (expressed in the original coordinate system on the left, and the new coordinate system on the right hand side) up to a boundary term provided by a specified but arbitrary function $F(q,Q,T)$:

$$S_P[C] \;=\; \int_C (p_i \frac{dq^i}{dt} - h(q,p,t))dt = \int_C (P_i \frac{dQ^i}{dT} - H(Q,P,T) + \frac{dF}{dT})dT \tag{F.7}$$

Expanding the right hand side:

$$\begin{aligned} &\int_C (P_i \frac{dQ^i}{dT} - H(Q,P,T) + \frac{\partial F}{\partial T} + \frac{\partial F}{\partial q^i}\frac{dq^i}{dT} + \frac{\partial F}{\partial Q^i}\frac{dQ^i}{dT})\frac{dT}{dt}dt \\ &= \int_C (P_i \frac{dQ^i}{dt} - H(Q,P,T)\frac{dT}{dt} + \frac{\partial F}{\partial t} + \frac{\partial F}{\partial q^i}\frac{dq^i}{dt} + \frac{\partial F}{\partial Q^i}\frac{dQ^i}{dt})dt \end{aligned} \tag{F.8}$$

If $\frac{\partial F}{\partial q^i} = p_i$, $\frac{\partial F}{\partial Q^i} = -P_i$, $H\frac{dT}{dt} - \frac{\partial F}{\partial t} = h$, then the right hand side of (F.7) is the same as the left hand side. The variational principle with fixed endpoints leads to the same geometrical trajectory in either coordinate systems. We assume for these equations that we can invert coordinate transformations. Expressing these equations in the original coordinates (q,p,t):

$$\frac{\partial F}{\partial q^i}(q, Q(q,p,t), t) = p_i, \tag{F.9}$$

$$\frac{\partial F}{\partial Q^i}(q, Q(q,p,t), t) = -P_i(q,p,t), \tag{F.10}$$

$$H(Q(q,p,t), P(q,p,t), T(t))\frac{dT}{dt}(t) - \frac{\partial F}{\partial t}(q, Q(q,p,t), t) = h(q,p,t) \tag{F.11}$$

These are the equations of a canonical transformation generated by $F = F(q,Q,T)$ or $F = F(q,Q,T(t))$.

Transformations with the property (F.7) can be obtained via any one of 4 types of generating functions

$$F_1(q,Q,t), F_2(q,P,t), F_3(p,Q,t), F_4(p,P,t) \tag{F.12}$$

together with $T(t)$, which are functions on time extended phase space defined using different coordinate systems where half the coordinates are from the old system z and half from the new Z: $F(z,Z,t)$. For regularity, we require the Hessian determinant to be non-zero (for invertibility of coordinate transformations), $\det(\frac{\partial^2 F_a}{\partial z_i \partial Z_j}) \neq 0$, for the generating functions F's we intend to employ, so that the coordinates span the region in extended phase space where we want to use the theory. Then we can obtain the canonical duals from the generating functions:

$$p_i = \frac{\partial F_1}{\partial q^i}(q,Q,t) \quad , \qquad P_i = -\frac{\partial F_1}{\partial Q^i}(q,Q,t), \tag{F.13}$$

$$p_i = \frac{\partial F_2}{\partial q^i}(q,P,t) \quad , \qquad Q^i = -\frac{\partial F_2}{\partial P_i}(q,P,t), \tag{F.14}$$

etc, and

$$H\frac{dT}{dt} = h + \frac{\partial F_a}{\partial t} \tag{F.15}$$

In the multiphase-space generalization it will be seen that only a type 1 generating function can be defined.

Appendix G

Hamilton-Jacobi theory in phase space

Hamilton-Jacobi Theory [45] is another formulation of classical mechanics, which generalizes to multisymplectic field theory. One reason it is of interest is because the Hamilton-Jacobi equation is the optical approximation to Schrödinger equation, and, in fact, led Schrodinger to his celebrated equation. Thus it forms a link between classical Hamiltonian mechanics and quantum mechanics.

In Hamilton-Jacobi theory, the equation of motion, called the *Hamilton-Jacobi* equation, is a single first order partial differential equation, instead of $2N$ first order or N second order ODEs as in Hamiltonian and Lagrangian mechanics respectively (with N configuration degrees of freedom $\bar{q} = q_1 \ldots q_N$). The complete set (parametrized by $\bar{P}$) of solutions to the Hamilton-Jacobi equation is the generating function $S(\bar{q}, t; \bar{P})$ of type 2 for a special type of canonical transformation, namely where the new coordinates $(\bar{Q}, \bar{P})$ are constant on trajectories and serve to parametrize solutions. The state $(\bar{q}, \bar{p})$ of the system in the original phase-space coordinates at time t is given by the transformation equations (F.14) between the coordinate systems. This generating function has also the property of being the action $S[q(t)]$ of a trajectory $q(t)$, depending only the endpoints. $S[\bar{q}(t)] = S(\bar{q}(t_f), t_f; \bar{P}) - S(\bar{q}(t_i), t_i; \bar{P})$

Another viewpoint for Hamilton-Jacobi theory has to do with the relation between phase space and configuration space, given a Hamiltonian function. Specifically, how to choose sections of extended phase space over time-extended configuration space, so that the Hamiltonian vector field lies tangent to the section - thereby foliating the section (and also the time-extended configuration space via the projection of the section) into trajectories. The function $S(q, t)$, with parameter $\bar{P}$ fixed, generates the section map (with $P_i = \frac{\partial S}{\partial q^i}(q,t;\bar{P})$) , and the Hamilton Jacobi equations ensure the tangency property above.

The Hamilton-Jacobi equation for the function $S(\bar{q},t)$ is:

$$\frac{\partial S}{\partial t} = -H(\bar{q}, \frac{\partial S}{\partial q^1}, \ldots, \frac{\partial S}{\partial q^N}, t) \tag{G.1}$$

where $H(\bar{q}, p_1, \ldots, p_N, t)$ is the Hamiltonian of the system and where the momenta in the Hamiltonian are substituted with the partial derivatives of $S(\bar{q},t)$ with respect to the configuration variables, to construct the right hand side of the equation. The solutions are $S(\bar{q},t;\bar{Q},t_0)$, where the $\bar{Q},t_0$ parametrize solutions, and on which the momenta p_i on a trajectory are $p_i(\bar{q},t) \approx \frac{\partial S}{\partial q^i}(\bar{q},t)$. Note that the Hamilton-Jacobi solutions are defined up to addition of an arbitrary constant.

G.1 The Hamilton-Jacobi map

We will now introduce Hamilton-Jacobi theory as a theory of maps, $T : \widetilde{Q} \longrightarrow \mathfrak{T}^*\widetilde{Q}$, from time-extended configuration space to extended phase space. This somewhat abstract approach brings out its symplectic-geometric character most clearly. This will have an immediate generalization to the multisymplectic Hamilton-Jacobi theory below.

Any function $S(\bar{q},t)$ on extended configuration space $\widetilde{Q}$ generates a section Γ_T of the extended phase-space bundle over extended configuration space $\pi_{\mathfrak{T}^*Q\times\mathbb{R},\tilde{Q}} : \mathfrak{T}^*\widetilde{Q} \longrightarrow \widetilde{Q}$, via the map

$$T : \tilde{Q} \longrightarrow \mathfrak{T}^*\widetilde{Q} :: (\bar{q},t) \longmapsto (t,\bar{q},p_i(\bar{q},t),s(\bar{q},t)) = (\ \ t,\bar{q}, \frac{\partial S}{\partial q^i}(\bar{q},t), \frac{\partial S}{\partial t}(\bar{q},t)\ \ ) \tag{G.2}$$

T may also be viewed as a 1-form on $\widetilde{Q}$:

$$T = p_i(\bar{q},t)\mathrm{d}q^i + s(\bar{q},t)\mathrm{d}t \tag{G.3}$$

T is in fact an exact 1-form on $\widetilde{Q}$:

$$T = p_i(\bar{q},t)\mathrm{d}q^i + s(\bar{q},t)\mathrm{d}t = \frac{\partial S}{\partial q^i}\mathrm{d}q^i + \frac{\partial S}{\partial t}\mathrm{d}t = \mathrm{d}S \tag{G.4}$$

T can also be viewed as the tautological form $\widetilde{T}$ in $\mathfrak{T}^*\widetilde{Q}$, defined on the section $T(\widetilde{Q})$, acting on the tangent space $\mathfrak{T}T(\widetilde{Q})$ of the embedding of $\Gamma_T = T(\widetilde{Q}) \subset \mathfrak{T}^*\widetilde{Q}$. $\widetilde{T}$ is in fact the canonical 1-form $\widetilde{\Theta}$ restricted to the subspace $\mathfrak{T}T(\widetilde{Q}) \subset \mathfrak{T}\mathfrak{T}^*\widetilde{Q}$, the tangent space to the embedded surface. The canonical 1-form $\widetilde{\Theta}$ pulled back to $\widetilde{Q}$ by the section map T is the form $T = \mathrm{d}S$ on $\widetilde{Q}$.

If, at a point on the section, we have two vectors $X, Y \in \mathfrak{T}T(\widetilde{Q}) \subset \mathfrak{T}\mathfrak{T}^*\widetilde{Q}$ lying in the section Γ_T, then their contraction with the symplectic form is zero:

$$X \lrcorner Y \lrcorner \widetilde{\Omega} = X \lrcorner Y \lrcorner \mathrm{d}\widetilde{\Theta} = \pi_*X \lrcorner \pi_*Y \lrcorner \mathrm{d}T = \pi_*X \lrcorner \pi_*Y \lrcorner \mathrm{d}\mathrm{d}S = 0 \tag{G.5}$$

The zero is a result of the closure of the form T.

This is the definition of an isotropic submanifold of a symplectic manifold. In fact $\Gamma_T = T(\widetilde{Q}) \subset \mathfrak{T}^*\widetilde{Q}$ is a lagrangian submanifold of $\mathfrak{T}^*\widetilde{Q}$ because its dimension, $N+1$, is half that of the ambient symplectic manifold $\mathfrak{T}^*\widetilde{Q}$ (and thus the maximal dimension of an isotropic submanifold, which is the definition of lagrangian submanifold). This desirable property arises from $\mathrm{d}T = 0$ so we could therefore generalize to T being closed rather than exact, and so $T = \mathrm{d}S$ on simply connected patches of $\widetilde{Q}$ rather than globally. Similarly $\mathrm{d}\widetilde{T} = 0$ so we can generalize to $\widetilde{T}$ being closed rather than exact, and so $\widetilde{T} = \mathrm{d}\widetilde{S}$ on simply connected patches of $T(\widetilde{Q})$ rather than globally. Conversely, a lagrangian section T of $\mathfrak{T}^*\widetilde{Q}$ corresponds to the existence of a local function S on $\widetilde{Q}$ such that $T = \mathrm{d}S$.

G.1.1 Integral of the 1-form T along a path in $\tilde{Q}$

The closed 1-form T can be integrated along any path $C_{\widetilde{Q}}$ in $\widetilde{Q}$, and gives the same result as the canonical 1-form $\widetilde{T} = p_i \mathrm{d}q^i + s\mathrm{d}t$ along the path $T(C_{\widetilde{Q}})$ (which is the path $C_{\widetilde{Q}}$ mapped pointwise to $\mathfrak{T}^*\widetilde{Q}$ by T). The integral is

$$\int_{C_{\widetilde{Q}}} T = \int_{C_{\widetilde{Q}}} \mathrm{d}S = S(\bar{q}_f, t_f) - S(\bar{q}_i, t_i) =: \Delta S = \int_{C_{\widetilde{Q}}} \frac{\partial S}{\partial q^i}\mathrm{d}q^i + \frac{\partial S}{\partial t}\mathrm{d}t$$

$$= \int (p_i(\bar{q}(t), t)dq^i + s(\bar{q}(t), t)dt) \llcorner \vec{v}_{C_{\widetilde{Q}}}\Big|_{C_{\widetilde{Q}}} = \int_{T(C_{\widetilde{Q}})} \widetilde{T} = \int (p_i dq^i + sdt) \llcorner \vec{v}_{T(C_{\widetilde{Q}})}\Big|_{T(C_{\widetilde{Q}})}$$

$$= \int_{T(C_{\widetilde{Q}})} \widetilde{\Theta} = S_{\widetilde{P}}[T(C_{\widetilde{Q}})] \tag{G.6}$$

which is the extended phase-space action of the path $T(C_{\widetilde{Q}})$ in extended phase space $\mathfrak{T}^*\widetilde{Q}$.

This action only depends on the endpoints in $\widetilde{Q}$ because we are restricting to paths in extended phase space determined by the map T, although the paths can be arbitrary in extended configuration space.

If we consider an infinitesimal variation $\delta_X(t)$ of the path $C_{\widetilde{Q}}$, considered as a vector field X along the path in $\widetilde{Q}$, extended to a vector field over a neighborhood of $C_{\widetilde{Q}}$ in $\widetilde{Q}$, then the change in the integral of the 1-form T along the path $C_{\widetilde{Q}}$ is

$$\delta_X \Delta S = \delta_X(S(\bar{q}_f, t_f) - S(\bar{q}_i, t_i)) = \delta_X \int_{C_{\widetilde{Q}}} T = \int_{C_{\widetilde{Q}}} \mathcal{L}_X T = \int_{C_{\widetilde{Q}}} (\mathrm{d}i_X + i_X \mathrm{d})T$$

$$= \int_{\partial C_{\widetilde{Q}}} i_X T + \int_{C_{\widetilde{Q}}} X \lrcorner \mathrm{d}\mathrm{d}S = \int_{\partial C_{\widetilde{Q}}} i_X \mathrm{d}S$$

$$= (S(\bar{q}_f + \delta q_f, t_f + \delta t_f) - S(\bar{q}_i + \delta q_i, t_i + \delta t_i)) - (S(\bar{q}_f, t_f) - S(\bar{q}_i, t_i))$$

$$= \int_{T(C_{\widetilde{Q}})} (\mathrm{d}i_{X^T} + i_{X^T}\mathrm{d})\widetilde{T} = \int_{\partial T(C_{\widetilde{Q}})} X^T \lrcorner (p_i \wedge dq^i + s \wedge \mathrm{d}t) + \int_{T(C_{\widetilde{Q}})} X^T \lrcorner (\mathrm{d}p_i \wedge dq^i + \mathrm{d}s \wedge \mathrm{d}t)$$

$$= \int_{\partial T(C_{\widetilde{Q}})} X^T \lrcorner \widetilde{\Theta} + \int_{T(C_{\widetilde{Q}})} X^T \lrcorner \widetilde{\Omega} = \int_{\partial T(C_{\widetilde{Q}})} X^T \lrcorner \widetilde{\Theta} + 0 = \int_{\partial T(C_{\widetilde{Q}})} X^T \lrcorner \widetilde{T} = \int_{\partial T(C_{\widetilde{Q}})} X^T \lrcorner \widetilde{\mathrm{d}S} \tag{G.7}$$

where X^T is the vector field X on $\widetilde{Q}$ pushed forward to $T(\widetilde{Q}) \subset \mathfrak{T}^*\widetilde{Q}$ by T. The path integral of $X^T \lrcorner \widetilde{\Omega}$ is zero in the last line above because both X^T and the line elements dl along the path lie in the isotropic submanifold Γ_T, and so, by the definition of 'isotropic' $X^T \lrcorner dl \lrcorner \widetilde{\Omega} = 0$. $\widetilde{\Omega}$ is the canonical symplectic form on $\mathfrak{T}^*\widetilde{Q}$.

The change in the extended phase-space action is $X^T \lrcorner \widetilde{\mathrm{d}S}$ at the end points only of the path $T(C_{\widetilde{Q}})$. This is as expected from the fact that the this action is a function of the end-points only.

G.2 Imposition of the Hamilton-Jacobi equation

Geometrically, the Hamilton-Jacobi equation is the hypersurface constraint $\bar{H} = 0$ imposed on T.

If $S(\bar{q}, t)$ is a solution of the Hamilton-Jacobi equation (G.1), $H \circ T = 0$, then $s = -H(\bar{q}, \bar{p}, t)$ and the integrals (G.6) above can be written

$$S(\bar{q}_f, t_f) - S(\bar{q}_i, t_i) = \int_{C_{\widetilde{Q}}} \frac{\partial S}{\partial q^i} \mathrm{d}q^i + \frac{\partial S}{\partial t} \mathrm{d}t = \int_{C_{\widetilde{Q}}} \mathrm{d}S = \int_{C_{\widetilde{Q}}} T =$$

$$= \int (p_i(\bar{q}(t), t) dq^i - H(\bar{q}(t), \bar{p}(\bar{q}(t)), t) dt) \llcorner \vec{v}_{C_{\widetilde{Q}}} \Big|_{C_{\widetilde{Q}}} = \int_{T(C_{\widetilde{Q}})} \widetilde{T} = \int (p_i dq^i - H(\bar{q}, \bar{p}, t) dt) \llcorner \vec{v}_{T(C_{\widetilde{Q}})} \Big|_{T(C_{\widetilde{Q}})}$$

$$= \int_{C_{\bar{H}}} \widetilde{\Theta} = \int (p_i dq^i + s dt) \llcorner \left(\frac{\partial}{\partial t} \right) \Big|_{C_{\bar{H}}} = \int_{C_{\mathfrak{T}^* Q}} (p_i \dot{q}^i - H) \, dt \tag{G.8}$$

which is the multiphase-space action of $T(C_{\widetilde{Q}})$.

Let S be an arbitrary function on the bundle of the configuration space over time, such that $\frac{\partial S}{\partial t}(\bar{q}, t) = -H(\bar{q}, \bar{p}(\bar{q}, t), t)$ and $\frac{\partial S}{\partial q^i}(\bar{q}, t) = p_i(\bar{q}, t)$ The infinitesimal change of S along any section of the configuration space bundle is $\mathrm{d}S = \frac{\partial S}{\partial t}(\bar{q}, t)\mathrm{d}t + \frac{\partial S}{\partial q^i}(\bar{q}, t)\mathrm{d}q^i = -H(\bar{q}, \bar{p}(\bar{q}, t), t)\mathrm{d}t + p_i(\bar{q}, t)\mathrm{d}q^i$. This is the change in the phase-space action on the path in phase space obtained by the mapping from the configuration space bundle to the extended phase space $\{(t, \bar{q}, \bar{p}, s)\}$, of the path in the configuration space, defined by the partial derivatives of the function $S(\bar{q}, t; \bar{P})$.

So far we have not assumed that the paths are trajectories. The additional condition (to the H-J equation) is Hamilton's equations for the time rate of change of the configuration variables.

$$\dot{q^i} = \frac{\partial H}{\partial p_i}(\bar{q}, \bar{p}(\bar{q}, t), t) = \frac{\partial H}{\partial p_i}(\bar{q}, T(\bar{q}, t), t) \tag{G.9}$$

Then $X = (\dot{\bar{q}}, \dot{t}) = (\frac{\partial H}{\partial \bar{p}}(\bar{q}, T(\bar{q}, t), 1)$ is the vector field of trajectories on $\widetilde{Q}$

If $C_{\mathfrak{T}^*Q}$ is a Hamiltonian trajectory, then the solutions S of the Hamilton-Jacobi equation measures (up to a constant) the action along the trajectory, as mentioned at the beginning of this section. $S(\bar{q}, t; \bar{P})$ is an observable in time-extended phase space which gives, at the point $(\bar{q}_1, \bar{P}_1, t_1)$, the action at that point of the particular Hamiltonian trajectory $(q = q(t), P = P_1(\text{constant}), t)$ which passes through that point. We are free to add an arbitrary function of P to the H-J generator: $S'(\bar{q}, t; \bar{P}) = S(\bar{q}, t; \bar{P}) + f(\bar{P})$, so as to reset the 'starting value' of the action, separately on all the P- labeled surfaces Γ_T, each of which is a lagrangian image of $T(\widetilde{Q})$, and which foliate time-extended phase space.

Curves which are solutions of Hamilton's equations will lie entirely in the section $T(\tilde{Q})$ or entirely outside it. Thus, in the regular case, the surface $T(\tilde{Q})$ of extended phase space is foliated by trajectories, and any foliation on $T(\tilde{Q})$ is projected down by T^{-1} to a foliation of the extended configuration space $\tilde{Q}$.

G.3 Hamilton-Jacobi theory and canonical transformations

The H-J equation can be viewed as a canonical transformation where the new Hamiltonian is a constant: The type 2 generating function $F(q, P, t)$ is chosen such that $h(q, p, t) + \frac{\partial F}{\partial t} = H = 0$ and $p = \frac{\partial F}{\partial q}$. Under Hamilton's equations the new canonical variables, $Q = \frac{\partial F}{\partial P}$ and P , are constant in time , therefore they label trajectories - effectively a new coordinatization of extended phase space compatible with trajectories, i.e. where the tangent vector to a trajectory is $\frac{\partial}{\partial t}$. The dynamical information is contained in the coordinate transformation functions: $q = q(Q, P, t), p = p(Q, P, t)$, which describes a trajectory labeled by a particular values $Q = Q_c, P = P_c$ and given by the transformation function directly as a function of t, $q = q(t; Q_c, P_c), p = p(t; Q_c, P_c)$.

The fact that the change in the generating function F between endpoints is the action along a trajectory can be seen from the integral (F.7) where, in this case, the only non zero term on the right hand side is $\int_C (\frac{dF}{dT}) dT = F(Q, P, T_f) - F(Q, P, T_i)$, where we used the fact that the new coordinates Q, P are constant along trajectories.

G.3.1 Hamilton-Jacobi theorem

This section has a close generalization in multiphase-space dynamics.

Let X_H be the Hamiltonian vector field in extended phase space corresponding to H, $X_H^T := \mathfrak{T}\pi^{\widetilde{Q},\mathfrak{T}^*\widetilde{Q}} \circ X_H \circ T$ this vector field on the subspace $T(\widetilde{Q})$ (but not necessarily *in* the subspace) vertically projected down to the base space $\widetilde{Q}$. If a vector field X on a subset of $\widetilde{Q}$ is such that $X = X_H^T$ on its domain, then it is said to be T-related to X_H. Let Y be the tangent vector along a path $(q(t), t)$ in $\widetilde{Q}$.

(Hamilton-Jacobi theorem) Then the following are equivalent:

(1) $\mathrm{d}H^{\widetilde{Q}}(q(t)) = 0$, where $H^{\widetilde{Q}}(q, t) := H \circ T(q, t)$ ($H^{\widetilde{Q}}$ is H pulled back to $\widetilde{Q}$)

(2) Y is T-related to X_H

(3) If $q(t)$ satisfies $\frac{\mathrm{d}q}{\mathrm{d}t} = \frac{\partial H}{\partial p}$ then $Tq(t)$ satisfies Hamilton's equations.

(4) If If $q(t)$ is an integral curve of X_H^Q then $Tq(t)$ is an integral curve of X_H (and the latter implies the Hamilton's equations). (So that integral curves on the projected vector field map to integral curves of the Hamiltonian vector field on the subspace $T\widetilde{Q}$.

Proof

(1) $\Rightarrow$ (3). We first relate $H^{\widetilde{Q}}$ in $\widetilde{Q}$ to H in $\mathfrak{T}^*\widetilde{Q}$:

$$\frac{\partial H^{\widetilde{Q}}}{\partial q^i}|_{\hat{q^i},t} = \frac{\partial H(q,p,t)}{\partial q^i}|_{\hat{q^i},p,t} + \frac{\partial H(q,p,t)}{\partial p^j}|_{q,\hat{p_j},t} \frac{\partial P_j(q,t)}{\partial q^i}|_{\hat{q^i},t} \tag{G.10}$$

Now we want to show HAM2: $\frac{\mathrm{d}p_i}{\mathrm{d}t} \approx \frac{\partial H(q,p,t)}{\partial q^i}|_{\hat{q^i},p,t} + \frac{\partial P_i(q,t)}{\partial t}|_q =: H2$. Starting with $H2$, and substituting for $\frac{\partial H(q,p,t)}{\partial q^i}|_{\hat{q^i},p,t}$ using (G.10), we obtain

$$\begin{aligned} H2 &= \frac{\partial H(q,p,t)}{\partial p^j}|_{q,\hat{p_j},t} \frac{\partial P_j(q,t)}{\partial q^i}|_{\hat{q^i},t} - \frac{\partial H^{\widetilde{Q}}}{\partial q^i}|_{\hat{q^i},t} + \frac{\partial P_i(q,t)}{\partial t}|_q \approx \\ &\approx \frac{\mathrm{d}q_i}{\mathrm{d}t} \frac{\partial P_j(q,t)}{\partial q^i}|_{\hat{q^i},t} - \frac{\partial H^{\widetilde{Q}}}{\partial q^i}|_{\hat{q^i},t} + \frac{\partial P_i(q,t)}{\partial t}|_q = \frac{\mathrm{d}P_i}{\mathrm{d}t} - \frac{\partial H^{\widetilde{Q}}}{\partial q^i}|_{\hat{q^i},t} \end{aligned} \tag{G.11}$$

Where in the second equality ($\approx$) we substituted HAM1: $\frac{\mathrm{d}q_i}{\mathrm{d}t} \approx \frac{\partial H(q,p,t)}{\partial p_i}|_{q,\hat{p}_i,t}$

Now (1) $\Rightarrow \frac{\partial H^{\widetilde{Q}}(q,t)}{\partial q^i}|_{\hat{q^i},t} = 0$, so we obtain $H2 \approx \frac{\mathrm{d}P_i}{\mathrm{d}t}$, which is HAM2 for $T(q(t), t)$. QED

So the non-closure of $\mathrm{d}H^{\widetilde{Q}}$ is the obstruction to force equation HAM2 and corresponds to an external force $\frac{\partial H^{\widetilde{Q}}(q,t)}{\partial q^i}|_{\hat{q^i},t}$.

(3) $\Rightarrow$ (1) if HAM1 and HAM2 hold, $H2 \approx \frac{\mathrm{d}P_i}{\mathrm{d}t} \Rightarrow \frac{\partial H^{\widetilde{Q}}(q,t)}{\partial q^i}|_{\hat{q^i},t} = 0 \Rightarrow \mathrm{d}H^{\widetilde{Q}} = 0 \Rightarrow$ (1). QED

(3) $\Leftrightarrow$ (2) If (3) Hamilton's equations hold for $T(q(t), t)$, with velocity vector T_*Y, then $T^*Y = \{\cdot, H\} = X_H$, the left hand equality is (2). QED

Appendix H

Hamilton-Jacobi Theory in multiphase space

The multisymplectic Hamilton-Jacobi theory which we examine below is concerned with parametrizing solutions to the DDW equations (3.48) and is the multiphase-space generalization of Hamilton-Jacobi theory in mechanics of appendix G. We also examine canonical transformations and see that only one type is defined, as opposed to four in phase space hamiltonian mechanics.

The multisymplectic Hamilton-Jacobi theory is a theory of a family of maps $\mathbb{T} : \mathcal{E} \longrightarrow \mathcal{M}$ which have the property that if the Γ is a section of $\mathcal{E}$, $\mathbb{T}(\mathcal{E})$ satisfies DDW2. Therefore, if $\mathbb{T}(\mathcal{E})$ satisfies DDW1, it automatically satisfies DDW2. In the case $d = 1$, this is Hamilton-Jacobi theory and $\mathbb{T}(\mathcal{E}) = \mathbb{T}(Q \times \mathbb{R})$ is foliated into trajectories, which are pulled back to $\mathcal{E}$ by $\mathbb{T}$, foliating $\mathcal{E}$. One way of viewing Hamilton-Jacobi theory is that it allows DDW1 and DDW2 to be treated separately. A second way of viewing Hamilton-Jacobi theory is that the $\mathbb{T}$'s parametrize families of solutions (the foliation) on $Q \times \mathbb{R}$. A third way of viewing Hamilton-Jacobi theory is that the $\mathbb{T}$'s have the property that $\mathbb{T} = \mathrm{d}S$, where S is a function on $Q \times \mathbb{R}$, which is in fact the action (up to a constant) of the trajectories of the foliation on $Q \times \mathbb{R}$.

The question of interest here is whether the foliation property can be reproduced in the multisymplectic Hamilton-Jacobi theory. If we could find a foliation of $\mathcal{E}$ into fields which satisfy DDW1, then the image $\mathbb{T}(\mathcal{E})$ will be foliated. If there is a foliation then the boundary values are foliated and therefore different for each leaf.

H.1 Theorem (generalized Hamilton-Jacobi)

This is described in [66] [41].

We define $\mathbb{T}$ to be a closed $d-2$ horizontal d-form on the configuration bundle $\mathcal{E}$.

Then, in a coordinate patch,

$$\mathbb{T} = P_i^\mu(u^i, x^\mu)\ \mathrm{d}u^i \wedge \mathrm{d}^{d-1}x_\mu + P(u^i, x^\mu)\ \mathrm{d}^d x \tag{H.1}$$

and

$$0 = \mathrm{d}\mathbb{T} = (\frac{\partial P}{\partial u^i} - \frac{\partial P_i^\mu}{\partial x^\mu})\ \mathrm{d}u^i \wedge \mathrm{d}^d x + (\frac{\partial P_j^\mu}{\partial u^i})\ \mathrm{d}u^i \wedge \mathrm{d}u^j \wedge \mathrm{d}^{d-1}x_\mu \tag{H.2}$$

so the coefficients on the RHS are zero:

$$\frac{\partial P}{\partial u^i} - \frac{\partial P_i^\mu}{\partial x^\mu} = 0 \quad \text{and} \quad \frac{\partial P_j^\mu}{\partial u^i} - \frac{\partial P_i^\mu}{\partial u^j} = 0 \tag{H.3}$$

Because of the identification of the multiphase space $\mathcal{M}$ with the space of $d-2$ horizontal d-forms on the configuration bundle $\mathcal{E}$, $\mathbb{T}$ can be viewed as a bundle map from the configuration bundle to the multiphase space: $T : \mathcal{E} \longrightarrow \mathcal{M} \quad :: \quad (u^i, x^\mu) \mapsto (u^i, x^\mu, p_i^\mu, p) = (x^\mu, u^i, P_i^\mu, P)$

Let $h^{\mathcal{E}}$ be a flat connection on the bundle $\pi^{\mathcal{E},B} : \mathcal{E} \longrightarrow B$, with $\Gamma = (x^\mu, u_\Gamma^i(x^\mu))$ an integral section of $\mathcal{E}$. Then $\frac{\partial u_\Gamma^i(x^\mu)}{\partial x^\mu}|_{u,\hat{x^\mu}} \approx \frac{\partial \mathcal{H}(u,p,x)}{\partial p_j^\mu}|_{u,\hat{p_j},x}$

We define the d-form

$$H^{\mathcal{E}}(u^i, x^\mu) := H \circ \mathbb{T}(u^i, x^\mu) = H(u^i, P_i^\mu(u^i, x^\mu), P(u^i, x^\mu), x^\mu)\ \mathrm{d}^d x$$

$$= (-\mathcal{H}(u^i, P_i^\mu(u^i, x^\mu), x^\mu) + P(u^i, x^\mu))\mathrm{d}^d x \tag{H.4}$$

which is the the extended DDW Hamiltonian d-form $H\mathrm{d}^d x$ on $T(\mathcal{E})$ pulled back to $\mathcal{E}$ via T.

We now calculate the exterior derivative of $H^{\mathcal{E}}$, the projection from $T(\mathcal{E})$ to $\mathcal{E}$ of the spacetime volume form $H\mathrm{d}^d x$. We will show that it is related to the DDW equations.

$$\mathrm{d}H^{\mathcal{E}} = \mathrm{d}\{\ H(u^i, P_i^\mu(u^i, x^\mu), x^\mu, P(u^i, x^\mu))\ \mathrm{d}^d x\ \}$$

$$= \ \frac{\partial}{\partial u^i}|_{\hat{u^i},x}\ \{\ (-\mathcal{H}(u^i, P_i^\mu(u^i, x^\mu), x^\mu) + P(u^i, x^\mu)\ \}\mathrm{d}u^i \wedge \mathrm{d}^d x$$

$$= -\left(\frac{\partial \mathcal{H}(u^i, p_i^\mu, x)}{\partial p_j^\mu}|_{u,\hat{p_j},x} \frac{\partial P_j^\mu(u,x)}{\partial u^i}|_{\hat{u^i},x} + \frac{\partial \mathcal{H}}{\partial u^i}|_{\hat{u^i},p,x} + \frac{\partial P(u,x)}{\partial u^i}|_{u,\hat{x^\mu}} \right) \mathrm{d}u^i \wedge \mathrm{d}^d x \tag{H.5}$$

Substituting (H.3) in the first and third term, we obtain:

$$\mathrm{d}H^{\mathcal{E}} = -\left(\frac{\partial \mathcal{H}(u^i, p_i^\mu, x)}{\partial p_j^\mu}|_{u,\hat{p_j},x} \frac{\partial P_i^\mu(u,x)}{\partial u^j}|_{\hat{u^j},x} + \frac{\partial \mathcal{H}}{\partial u^i}|_{\hat{u^i},p,x} + \frac{\partial P_i^\mu(u,x)}{\partial x^\mu}|_{u,\hat{x^\mu}} \right) \mathrm{d}u^i \wedge \mathrm{d}^d x \tag{H.6}$$

If we consider a section $u^i_\Gamma(x^\mu)$ of $\mathcal{E}$, and the corresponding momenta are given by $T(u^i_\Gamma(x^\mu), x^\mu)$, we write the difference from DDW1:

$$E^j_\mu(x^\mu) := \frac{\partial u^j_\Gamma(x^\mu)}{\partial x^\mu}|_{\hat{x^\mu}} - \frac{\partial \mathcal{H}(u,p,x)}{\partial p^\mu_j}|_{u,\hat{p_j},x} \; (u^i_\Gamma(x^\mu), P^\mu_i(u^i_\Gamma(x^\mu), x^\mu), P(u^i_\Gamma(x^\mu), x^\mu), x^\mu) \tag{H.7}$$

Using this we can substitute for $\frac{\partial \mathcal{H}(u,p,x)}{\partial p^\mu_j}$ in the first term of (H.6) to obtain

$$\mathrm{d}H^{\mathcal{E}} =$$

$$-\left(\frac{\partial u^j_\Gamma(x^\mu)}{\partial x^\mu}|_{\hat{x^\mu}} \frac{\partial P_i^\mu(u,x)}{\partial u^j}|_{\hat{u^j},x} + \frac{\partial \mathcal{H}}{\partial u^i}|_{\hat{u^i},p,x} + \frac{\partial P_i^\mu(u,x)}{\partial x^\mu}|_{u,\hat{x^\mu}} - E^j_\mu(x^\mu) \frac{\partial P_i^\mu(u,x)}{\partial u^j}|_{\hat{u^j},x} \right) \mathrm{d}u^i \wedge \mathrm{d}^d x \tag{H.8}$$

This is only defined on the points of the section Γ of $\mathcal{E}$.

Now,

$$\frac{\partial u^j_\Gamma(x^\mu)}{\partial x^\mu}|_{\hat{x^\mu}} \frac{\partial P_i^\mu(u,x)}{\partial u^j}|_{\hat{u^j},x} + \frac{\partial P_i^\mu(u,x)}{\partial x^\mu}|_{u,\hat{x^\mu}} + \frac{\partial \mathcal{H}}{\partial u^i}|_{\hat{u^i},p,x} = \frac{\partial P_i^\mu(x)}{\partial x^\mu}|_{\hat{x^\mu}} + \frac{\partial \mathcal{H}}{\partial u^i}|_{\hat{u^i},p,x} =: -E_i(x^\mu) \tag{H.9}$$

which is the difference from DDW2 of a field configuration, where the multimomenta are given by $T(u^i_\Gamma(x^\mu), x^\mu)$. Substituting into (H.8), we obtain

$$\mathrm{d}H^{\mathcal{E}} = \left(E_i(x^\mu) + E^j_\mu \frac{\partial P_i^\mu(u,x)}{\partial u^j}|_{\hat{u^j},x} \right) \mathrm{d}u^i \wedge \mathrm{d}^d x \; =$$

$$= \left(\frac{\partial S_{MP}(u^i_\Gamma, p^\mu_i, x)}{\partial u^i}|_{\hat{u^i},p,x}(x^\mu) + \frac{\partial S_{MP}(u^i_\Gamma, P^\mu_i, x)}{\partial P^\mu_i}|_{u^i,\hat{P}^\mu_i,x}(x^\mu) \frac{\partial P^\mu_j(u,x)}{\partial u^i}|_{\hat{u^j},x} \right) \mathrm{d}u^i \wedge \mathrm{d}^d x \tag{H.10}$$

here we used (H.3) again. The variation of the multiphase-space action is from (3.51), which employs integration by parts, so we assume there is no variation of the u^i on the boundary of the region of integration.

We finally obtain,

$$\mathrm{d}H^{\mathcal{E}} \;=\; \left(\frac{\partial S_{MP}(u^i_\Gamma, P^\mu_i(u^i, x^\mu), x)}{\partial u^i}|_{\hat{u^i},p,x}(x^\mu) \right) \mathrm{d}u^i \wedge \mathrm{d}^d x \tag{H.11}$$

which is defined on the field configuration $\Gamma \subset \mathcal{E}$. If $\delta u(x^\mu)$ is an infinitesimal variation of the path $u^i_\Gamma(x^\mu)$, then (H.11) says that $\delta u(x^\mu) \lrcorner \mathrm{d}H^{\mathcal{E}}$, integrated over the field configuration $\Gamma \subset \mathcal{E}$, is equal the corresponding variation δS_{MP} of the multiphase-space action of $T(u^i_\Gamma(x^\mu), x^\mu)$. The variation of the path $\delta u^j(x^\mu)$ is assumed to be zero on the boundary of the region of integration. The multimomenta (which contribute to the multiphase-space action) are determined by the mapping $T(u^i_\Gamma(x^\mu), x^\mu)$.

We now consider the case that $T(u^i_\Gamma(x^\mu), x^\mu)$ is a trajectory, i.e. the DDW equations hold for $T(u^i_\Gamma(x^\mu), x^\mu)$. In this case $E_i(x^\mu) \approx 0 \approx \overset{j}{E}_\mu(x^\mu)$ and so $\mathrm{d}H^{\mathcal{E}} \approx 0$ on Γ. So an arbitrary infinitesimal variation of $u^i_\Gamma(x^\mu)$ results in no (first order) variation in the multiphase-space action (which is equal to the classical action) $S_{MP}[T(u^i_\Gamma(x^\mu), x^\mu)] = S[u^i_\Gamma(x^\mu)]$.

The following are equivalent:

(1) If DDW1 holds for the section $\Gamma = (u^i_\Gamma(x^\mu), x^\mu)$:

$$\frac{\partial u^i_\Gamma(x^\mu)}{\partial x^\mu}|_{u,\hat{x^\mu}} - \frac{\partial \mathcal{H}(u,p,x)}{\partial p^\mu_j}|_{u,\hat{p_j},x}(u^i_\Gamma(x^\mu), P^\mu_i(u^i_\Gamma(x^\mu), x^\mu), P(u^i_\Gamma(x^\mu), x^\mu), x^\mu) \approx 0 \quad \text{(H.12)}$$

then $T(\Gamma)$ also satisfies DDW2, and so $u^i_\Gamma(x^\mu)$ satisfies the equations of motion.

(2) $H^{\mathcal{E}}$ is closed, $\mathrm{d}H^{\mathcal{E}} \approx 0$, on $\mathcal{E}$.

Proof (1) $\Rightarrow$ (2): This was shown immediately above

(2) $\Rightarrow$ (1): (2) $\Rightarrow$ (for a section Γ) $E_i(x^\mu) + \overset{j}{E}_\mu \frac{\partial P^\mu_i(u,x)}{\partial u^j}|_{\hat{u^j},x} = 0$. But this implies that $\overset{j}{E}_\mu = 0$ for $\mu = 0 \ldots d-1, i = 1 \ldots N \Rightarrow E_i(x^\mu) = 0$, for $i = 1 \ldots N$. This last is (1).

H.2 Solutions of DeDonder-Weyl equations

We next consider the relation of the distribution defined by the vectors Y_μ in (E.7) and the DeDonder-Weyl equations with Hamiltonian $\mathcal{H}$.

A solution of the DeDonder-Weyl equations with the DDW Hamiltonian $\mathcal{H}(x^\mu, u^i, p^\mu_i)$, given by smooth functions, $u^i(x^\mu), p^\mu_i(x^\mu)$, on spacetime B:

$$\frac{\partial u^i}{\partial x^\mu} \approx \frac{\partial \mathcal{H}}{\partial p^\mu_i} \quad , \quad \frac{\partial p^\mu_i}{\partial x^\mu} \approx -\frac{\partial \mathcal{H}}{\partial u^i} \quad \text{(H.13)}$$

together with

$$p(x^\mu) = -\mathcal{H}(x^\mu, u^i(x^\mu), p^\mu_i(x^\mu)) \quad \text{(H.14)}$$

defines a section $\Gamma = \{ \ (\ (x^\mu), u^i(x^\mu), p^\mu_i(x^\mu), p(x^\mu) \ ) \ \}$ of the bundle $\pi_{\mathcal{M},B} : \mathcal{M} \longrightarrow B$ whose embedding in $\mathcal{M}$ has an d-dimensional tangent space which can be expressed by the n-multivector Y_H defined by (E.7), for some functions $A^\nu_{\mu i}(x, u^i, p^\mu_i, p)$, and where H is defined by (E.1), on the points of $\Gamma \subset \mathcal{M}$. Thus the tangent multivector Y_H to Γ is Hamiltonian for H, and Γ lies inside the hypersurface $H = 0$.

H.3 Integrability and the generalized Hamilton-Jacobi equations

Conversely, we now consider the conditions for the distribution defined by Y_H, constructed from (E.7), to be integrable, leading to a surface, or families of surfaces (foliation) which are solutions of the DDW equations in the sense above.

We now seek smooth sections $\mathbb{T} : \mathcal{E} \longrightarrow \mathcal{M}$ where the multivector distribution Y_H in the surface $\mathbb{T}(\mathcal{E}) \subset \mathcal{M}$ integrate to a foliation $\mathcal{F}$ of $\mathbb{T}(\mathcal{E})$ in such a way that the leaves, $\mathbb{T}(\Gamma^{\mathcal{F}})$, of this foliation are solutions of the DDW equations. The projection of $\mathbb{T}(\mathcal{E})$ onto $\mathcal{E}$ is bijective and foliates $\mathcal{E}$ into leaves $\Gamma^{\mathcal{F}}$ which are field configuration solutions.

For a regular (i.e. from a non degenerate Legendre transformation) DDW Hamiltonian, any solution of the DeDonder-Weyl equations will be surfaces in $\mathcal{M}$ which correspond to local foliations on the configuration space $\mathcal{E}$ mapped to $\mathcal{M}$ by any mapping $\mathbb{T} : \mathcal{E} \longrightarrow \mathcal{M} \cong \Lambda^n_1\mathcal{E} :: (x^\mu, u^i) \longmapsto (x, u^i, p^\mu_i, p) = (x^\mu, u^i, T^\mu_i, T_p)$, which is a section of the bundle of $d-1$-horizontal d-forms. The map $\mathbb{T}$ may be viewed as an $d-1$-horizontal d-form on $\mathcal{E}$, $\mathbb{T}(x^\mu, u^i) = T^\mu_i(x^\mu, u^i)\ \mathrm{d}u^i \wedge \mathrm{d}_\mu x + T_p(x^\mu, u^i)\ \mathrm{d}^n x$. In addition we need the condition that the form $\mathbb{T}$ satisfies the equations $\mathrm{d}\mathbb{T} = 0$ and $\mathrm{d}(H \circ \mathbb{T}) = 0$. The closure condition on the form $\mathbb{T}$ implies that, on a simply connected patch of $\mathcal{E}$, $\mathbb{T} = \mathrm{d}S = \partial_i S^\mu\ \mathrm{d}u^i \wedge \mathrm{d}_\mu x + \partial_\mu S^\mu\ \mathrm{d}^d x$, for some $d-1$ -horizontal $d-1$ -form $S = S^\mu(x^\mu, u^i)\ \mathrm{d}x_\mu$, called *Hamilton's $d-1$-form*. Note that an arbitrary closed $d-1$-form may be added to S without changing $\mathbb{T}$. We now consider the tautological n-form $\widetilde{\Theta}$ on $\mathcal{M} \cong \Lambda^n_1\mathcal{E}$. The mapping $\mathbb{T} : \mathcal{E} \longrightarrow \mathcal{M} \cong \Lambda^n_1\mathcal{E} :: (x^\mu, u^i) \longmapsto (x, u^i, p^\mu_i, p) = (x^\mu, u^i, T^\mu_i, T_p)$ pulls back the tautological form of $\Lambda^n_1\mathcal{E}$ on $\mathbb{T}(\mathcal{E})$ to $\mathbb{T}$ viewed as a form on $\mathcal{E}$: $\mathbb{T}^*\widetilde{\Theta} = \mathbb{T}$. Similarly the bundle projection acting on $\mathbb{T}(\mathcal{E})$ pulls back the form $\mathbb{T}$ on $\mathcal{E}$ to the tautological form $\widetilde{\Theta}|_{\mathbb{T}(\mathcal{E})} =: \widetilde{\mathbb{T}}$ on $\Lambda^n_1\mathcal{E}$ restricted to $\mathbb{T}(\mathcal{E})$: $\pi^*\mathbb{T} = \widetilde{\Theta}|_{\mathbb{T}(\mathcal{E})}$. The bundle projection acting on $\mathbb{T}(\mathcal{E})$ pulls back Hamilton's $d-1$ form S on $\mathcal{E}$ to the form $\widetilde{S}$ on $\mathbb{T}(\mathcal{E}) \subset \Lambda^n_1\mathcal{E}$. As a result, $\widetilde{\Theta}|_{\mathfrak{T}(\mathbb{T}(\mathcal{E}))} = \mathrm{d}\widetilde{S}$. $\widetilde{\Omega} := -\mathrm{d}\widetilde{\Theta}$ is the multisymplectic form on $\mathcal{M}$ and so the multisymplectic form restricted to $\mathfrak{T}(\mathbb{T}(\mathcal{E}))$ is: $\widetilde{\Omega} := -\mathrm{d}\widetilde{\Theta} = -\mathrm{d}\mathrm{d}\widetilde{S} = 0$. We call such a submanifold of a multisymplectic manifold 'isotropic'. This is the same as the definition of an isotropic submanifold of a symplectic manifold.

The condition $\mathrm{d}(H \circ \mathbb{T}) = 0$ ensures that $-H(x^\mu, u^i, p^\mu_i, p) = p + \mathcal{H}(x^\mu, u^i, p^\mu_i)$ is constant on $\mathbb{T}(\mathcal{E})$ and therefore on the leaves. Setting this constant to zero, and using $\mathbb{T} = p^\mu_i\ \mathrm{d}u^i \wedge \mathrm{d}_\mu x + p\ \mathrm{d}^d x = \mathrm{d}S = \partial_i S^\mu\ \mathrm{d}u^i \wedge \mathrm{d}_\mu x + \partial_\mu S^\mu\ \mathrm{d}^d x$, we have

$$0 = -H = p + \mathcal{H} = \partial_\mu S^\mu + \mathcal{H}(x^\mu, u^i, \partial_i S^\mu) \tag{H.15}$$

which are the generalized Hamilton-Jacobi equations.

Combining the two conditions on $\mathbb{T}$, we obtain

$$0 = \mathrm{d}(H \circ \mathbb{T}) = (-\partial_\mu T^\mu_i + \partial_i T_p)\ \mathrm{d}u^i \wedge \mathrm{d}^d x$$
$$\approx -(\partial_\mu p^\mu_i(u^i, x^\nu) + \partial_i \mathcal{H}(u^i, T^\nu_i(u^i, x^\nu), x^\nu)\ \mathrm{d}u^i \wedge \mathrm{d}^d x \tag{H.16}$$

which are the DDW2 equations, $\partial_\mu p^\mu_i + \partial_i \mathcal{H} \approx 0$. Here the DDW2 equations hold on the entire space $\mathbb{T}(\mathcal{E})$, rather than on a particular path $\mathbb{T}(u^i_\Gamma(x^\nu))$. The DDW1 equation of motion

$$\partial_\mu u^i_\Gamma(x^\nu) - \partial^i_\mu \mathcal{H}(u^i_\Gamma(x^\nu), T^\nu_i(u^i_\Gamma(x^\nu), x^\nu), x^\nu) \approx 0 \tag{H.17}$$

are additional conditions which need to be imposed separately on individual field configurations $u^i_\Gamma(x^\nu)$ (sections Γ of the bundle $\pi^{B,\mathcal{E}} : \mathcal{E} \longrightarrow B$), so that $\mathbb{T}(u^i_\Gamma(x^\nu))$ satisfy the DDW equations.

To show solutions of the DDW equations foliate $\mathbb{T}(\mathcal{E})$, we need to show solutions the DDW1 equation foliate $\mathcal{E}$. The mapping $\mathbb{T}$, with the two conditions $\mathrm{d}\mathbb{T} = 0$ and $\mathrm{d}(H \circ \mathbb{T}) = 0$, ensures that DDW2 is satisfied.

H.4 The integral of $\mathbb{T}$ along a configuration in $\mathcal{E}$

The form field $\mathbb{T}$ on $\mathcal{E}$ can be integrated over any d-dimensional section $\Gamma_\mathcal{E}$ in $\mathcal{E}$, and producing the same result as the canonical d-form $\widetilde{\mathbb{T}} = p^\mu_i \mathrm{d}u^i \wedge \mathrm{d}^{d-1}x_\mu + p\mathrm{d}^d x$ integrated along the multiphase space $\mathcal{M}$ (viewed as a bundle over B) section $\Gamma_{T(\mathcal{E})} := T(\Gamma_\mathcal{E})$ which is the section $\Gamma_\mathcal{E}$ mapped pointwise to $\mathcal{M}$ by T.

The integral is

$$\Delta S := \int_{\Gamma_\mathcal{E}} \mathbb{T} = \int_{\Gamma_\mathcal{E}} \mathrm{d}(S^\mu \mathrm{d}^{d-1}x_\mu) = \int_{\Gamma_\mathcal{E}} \frac{\partial S^\mu}{\partial u^i}\mathrm{d}u^i \wedge \mathrm{d}^d x + \frac{\partial S^\mu}{\partial x^\mu}\mathrm{d}^d x \ =$$
$$\int_{\partial\Gamma_\mathcal{E}} S^\mu \mathrm{d}^{d-1}x_\mu = \int_{\partial\Gamma_\mathcal{E}} S^\mu\, dn_\mu = \int_{\Gamma_\mathcal{E}} (p^\mu_i(\bar{u}(x), x)\mathrm{d}u^i \wedge \mathrm{d}^{n-1}x_\mu + p(\bar{u}(x), x)\mathrm{d}^d x) \llcorner \vec{Y}_{\Gamma_\mathcal{E}}\Big|_{\Gamma_\mathcal{E}} (\bar{u}(x), x)$$
$$= \int_{T(\Gamma_\mathcal{E})} \widetilde{\mathbb{T}} = \int (p^\mu_i \mathrm{d}u^i \wedge \mathrm{d}^{d-1}x_\mu + p\mathrm{d}^d x) \llcorner \vec{Y}_{T(\Gamma_\mathcal{E})}\Big|_{T(\Gamma_\mathcal{E})} (x, \bar{u}(x), p^\mu_i, p) = S_{T(\Gamma_\mathcal{E})} \tag{H.18}$$

where, in the last two lines, the integration is performed using $\vec{Y}_{\Gamma_\mathcal{E}}|_{\Gamma_\mathcal{E}}$, the multivector elements of the surface $\Gamma_\mathcal{E}$, and similarly over $\vec{Y}_{T(\Gamma_\mathcal{E})}|_{T(\Gamma_\mathcal{E})}$, the multivector elements of the surface $T(\Gamma_\mathcal{E})$. n_μ is the normal to the surface $\partial\Gamma_\mathcal{E}$, which is the boundary of the integration region $\Gamma_\mathcal{E}$.

$S_{T(\Gamma_\mathcal{E})}$ is the multiphase-space action of the multiphase-space configuration $T(\Gamma_\mathcal{E})$ in multiphase space $\mathcal{M}$. Because $\widetilde{T} = \widetilde{\Theta}|_{\mathfrak{T}(T(\mathcal{E}))}$, the canonical d-form (restricted to $\mathfrak{T}(T(\mathcal{E}))$), also called the tautological form on $\mathcal{M}$. The multiphase-space action $S_{T(\Gamma_\mathcal{E})}$ of the multiphase-space configuration $T(\Gamma_\mathcal{E})$ only depends on the boundary values of $\mathbb{T}$ because of the closure of $\mathbb{T}$: $\mathrm{d}\mathbb{T} = 0$, which allows $\mathbb{T} = \mathrm{d}S$ on a simply connected region of $\mathcal{E}$, for some $d-1$-form S.

Variations $\delta_X(x)$ of the configuration

If we consider an infinitesimal variation $\delta_X(x)$ of the configuration $\Gamma_{\mathcal{E}}$, considered as a vector field X on the section $\Gamma_{\mathcal{E}}$ in $\mathcal{E}$, extended to a vector field over all of $\mathcal{E}$, then the change in the integral is

$$\delta_X \Delta S = \delta_X \int_{\Gamma_{\mathcal{E}}} \mathbb{T} = \int_{\Gamma_{\mathcal{E}}} \mathcal{L}_X \mathbb{T} = \int_{\Gamma_{\mathcal{E}}} (\mathrm{d} i_X + i_X \lrcorner\, \mathrm{d})\ \mathrm{d}(S^\mu \mathrm{d}^{d-1} x_\mu) = \int_{\partial \Gamma_{\mathcal{E}}} i_X \mathbb{T}\ +\ 0$$

$$= \int_{\partial \Gamma_{\mathcal{E}}} \delta_X S^\mu\ \mathrm{d}^{d-1} n_\mu \tag{H.19}$$

The variation only depends on the variation at the boundary. The integral of the form field $\mathbb{T}$ on a section $\Gamma_{\mathcal{E}}$ is constant under continuous deformations of the configurations $\Gamma_{\mathcal{E}}$ in $\mathcal{E}$, where the boundary and boundary values are kept fixed on $\partial \Gamma_{\mathcal{E}}$.

$$\delta_X \Delta S = \int_{\mathbb{T}\Gamma_{\mathcal{E}}} (\mathrm{d} i_{X^T} + i_{X^T} \mathrm{d}) \widetilde{\mathbb{T}}$$

$$= \int_{\partial \mathbb{T}\Gamma_{\mathcal{E}}} X^T \lrcorner\, (p_i^\mu \mathrm{d}u^i \wedge \mathrm{d}^{d-1} x_\mu + p \mathrm{d}^d x) \llcorner\, \vec{Y}_{T(\Gamma_{\mathcal{E}})}\Big|_{T(\Gamma_{\mathcal{E}})}$$

$$+ \int_{\mathbb{T}\Gamma_{\mathcal{E}}} X^T \lrcorner\, (\mathrm{d}p_i^\mu \wedge \mathrm{d}u^i \wedge \mathrm{d}^{d-1} x_\mu + \mathrm{d}p \wedge \mathrm{d}^d x) \llcorner\, \vec{Y}_{T(\Gamma_{\mathcal{E}})}\Big|_{T(\Gamma_{\mathcal{E}})}$$

$$= \int_{\partial \mathbb{T}\Gamma_{\mathcal{E}}} X^T \lrcorner\, \widetilde{\Theta} + \int_{\mathbb{T}\Gamma_{\mathcal{E}}} X^T \lrcorner\, \widetilde{\Omega} = \int_{\partial \mathbb{T}\Gamma_{\mathcal{E}}} X^T \lrcorner\, \widetilde{\mathbb{T}} = \delta_X S_{T(\Gamma_{\mathcal{E}})} \tag{H.20}$$

where X^T is the vector field X on $\mathcal{E}$ pushed forward to $\mathbb{T}\Gamma_{\mathcal{E}} \subset \mathcal{M}$ by $\mathbb{T}$. The path integral of $X^T \lrcorner\, \widetilde{\Omega}$ is zero in the above because both X^T and the multivector elements Y along the configuration lie in the submanifold $\mathbb{T}\Gamma_{\mathcal{E}} \subset \mathbb{T}(\mathcal{E})$ and for $\mathfrak{T}(\mathbb{T}(\mathcal{E}))$, $\widetilde{\Omega} = \mathrm{d}\widetilde{\Theta} = \mathrm{d}\widetilde{\mathbb{T}} = \mathrm{d}\mathrm{d}\widetilde{S} = 0$, the isotropic property of $\mathbb{T}(\mathcal{E}) \subset \mathcal{M}$. $\widetilde{\Omega}$ is the multisymplectic form on $\mathcal{M}$.

The variation only depends on the variation at the boundary. Note that the variation in multiphase space is limited to those tangent to the surface $T(\mathcal{E})$ in multiphase space $\mathcal{M}$ and are not general variations away from $T(\mathcal{E})$. The variation in the multimomenta and energy are determined by the variation in the basic fields and spacetime. The integral of the form field $\mathbb{T}$ on a section $\Gamma_{\mathcal{E}}$ is constant under continuous deformations of the configurations $\Gamma_{\mathcal{E}}$ in $\mathcal{E}$, where the boundary $\partial \Gamma_{\mathcal{E}}$ and boundary values are kept fixed.

Impose multiphase-space Hamilton-Jacobi equation

If $S^\mu(\bar{q}, t)$ is, in addition, a solution of the generalized Hamilton-Jacobi equation (H.15), $H \circ T = 0$, then $p = -\mathcal{H}(\bar{q}, \bar{p}, t)$ and the integrals (H.18) above may be written

$$\Delta S = \int \mathrm{d}S = \int \mathbb{T} = \int_{\Gamma_{\mathcal{E}}} (p_i^\mu(\bar{u}(x), x) \mathrm{d}u^i \wedge \mathrm{d}^{n-1} x_\mu\ -\ \mathcal{H}(\bar{u}(x), x) \mathrm{d}^d x) \llcorner\, \vec{Y}_{\Gamma_{\mathcal{E}}}\Big|_{\Gamma_{\mathcal{E}}} (\bar{u}(x), x)$$

$$= \int_{T(\Gamma_{\mathcal{E}})} \widetilde{\mathbb{T}} = \int (p_i^\mu \mathrm{d}u^i \wedge \mathrm{d}^{d-1} x_\mu\ -\ \mathcal{H} \mathrm{d}^d x) \llcorner\, \vec{Y}_{T(\Gamma_{\mathcal{E}})}\Big|_{T(\Gamma_{\mathcal{E}})} (x, \bar{u}(x), p_i^\mu, p) = S_{T(\Gamma_{\mathcal{E}})} \tag{H.21}$$

H.4.1 Change of variables: method of generating functions

We want new variables $U^k = U^k(u,p,x), P_k^\nu = P_k^\nu(u,p,x), X^\nu = X^\nu(x)$ on the multiphase-space bundle over spacetime. The DDW Hamiltonian $H = H(U,P,X)$ should be such that it has the same (up to a multiplicative constant C) multiphase-space action along any section $\Gamma_{\mathcal{M}}$, up to a boundary term:

$$C\int_{\Gamma_{\mathcal{M}}}(p_i^\mu\partial_\mu u^i - h(u,p,x))d^dx = \int_{\Gamma_{\mathcal{M}}}(P_k^\nu\partial_\nu U^k - H(U,P,X) + \partial_\nu F^\nu)d^dX \tag{H.22}$$

Expanding the right hand side, considering F to be a function of (u,U,X) and constant in the remaining unspecified coordinates (which we might take to be the spatial multimomenta P_k^s or p_i^s) :

$$\int_{\Gamma_{\mathcal{M}}}(P_k^\lambda\partial_\lambda U^k - H(U,P,X) + \partial_\nu F^\nu)d^dX =$$
$$= \int_{\Gamma_{\mathcal{M}}}(P_k^\lambda\partial_\lambda U^k - H(U,P,X) + \frac{\partial F^\lambda}{\partial X^\lambda} + \frac{\partial F^\lambda}{\partial u^i}\frac{\partial u^i}{\partial X^\lambda} + \frac{\partial F^\lambda}{\partial U^k}\frac{\partial U^k}{\partial X^\lambda})\ Det(\frac{\partial X^\kappa}{\partial x^\lambda})d^dx$$
$$= \int_{\Gamma_{\mathcal{M}}}(\frac{\partial x^\mu}{\partial X^\lambda}P_k^\lambda\partial_\mu U^k -\ Det(\frac{\partial X^\kappa}{\partial x^\lambda})H(U,P,X(x)) + \frac{\partial F^\mu}{\partial x^\mu} + \frac{\partial F^\mu}{\partial u^i}\partial_\mu u^i + \frac{\partial F^\mu}{\partial U^k}\partial_\mu U^k)d^dx \tag{H.23}$$

The notation $\partial_\nu F^\lambda$ is employed to indicate the derivative of the function F^λ of $\mathcal{M}$ in the direction ν (which projects to the coordinate line of X^ν on the spacetime base space) along the section $\Gamma_{\mathcal{M}}$, which is a particular multiphase-space field configuration. In contrast, the notation $\frac{\partial F^\mu}{\partial x^\mu}$ is used to indicate the derivative of the function F^λ of $\mathcal{M}$ in the direction ν along the coordinate line x_μ with the other coordinates being held fixed (although there is some latitude because F is constant along the unspecified coordinates), and is independent of the section $\Gamma_{\mathcal{M}}$, except for the fact that it is evaluated on $\Gamma_{\mathcal{M}}$, because it occurs in the integrand of an integral evaluated on $\Gamma_{\mathcal{M}}$.

If $Cp_i^\mu = \frac{\partial F^\mu}{\partial u^i}$, $\frac{\partial F^\mu}{\partial U^j} = \frac{\partial x^\mu}{\partial X^\lambda}P_i^\lambda$, $Det(\frac{\partial X^\kappa}{\partial x^\lambda})H - \frac{\partial F^\mu}{\partial x^\mu} = Ch$, then the variational principle with fixed endpoints leads to the same trajectory in both coordinate systems. We assume for these equations that we can invert coordinate transformations. Note that the equations for H and P_i^μ involve the Jacobian because H and P_i^μ are tensor densities. Expressing these equations in the coordinates (u,p,x):

$$Cp_i^\mu = \frac{\partial F^\mu}{\partial u^i}(u,U(u,p,x),x), \tag{H.24}$$

$$\frac{\partial F^\mu}{\partial U^k}(u,U(u,p,x),x) = \frac{\partial x^\mu}{\partial X^\lambda}(x)P_k^\lambda(u,p,x), \tag{H.25}$$

$$Det(\frac{\partial X^\kappa}{\partial x^\lambda})(x)H(U(u,p,x),P(u,p,x),X(x)) - \frac{\partial F^\mu}{\partial x^\mu}(u,U(u,p,x),x) = Ch(u,p,x) \tag{H.26}$$

These are the equations of a canonical transformation generated by $F^\nu = F^\nu(u,U,X)$ or $F^\nu = F^\nu(u,U,X(x))$.

Note that if we seek F so that $H = 0$ then in the new coordinates the DDW equations of motion are $U^k \approx$ constant and $\partial_\lambda P_k^\lambda = Y_h \lrcorner \mathrm{d}(P_k^\nu \mathrm{d}X_\nu) \approx 0$. The new coordinates for the multiphase-space bundle $\mathcal{M}$, U^k , P_k^ν, X^ν are such that there is a solutions which lies in the surface $U = U_0$ for every constant U_0 and for every divergence free P_k.

The equations above become

$$Cp_i^\mu = \frac{\partial F^\mu}{\partial u^i}(u, U_0, x) \tag{H.27}$$

$$\frac{\partial F^\mu}{\partial U^k}(u, U_0, x) = \frac{\partial x^\mu}{\partial X^\lambda}(x) P_k^\lambda(u, p, x) \tag{H.28}$$

$$-\frac{\partial F^\mu}{\partial x^\mu}(u, U_0, x) = Ch(u, p, x) \tag{H.29}$$

Combining the above, to eliminate p_i^μ:

$$-\frac{\partial F^\mu}{\partial x^\mu}(u, U_0, x) = h(u, \frac{\partial F^\mu}{\partial u^i}(u, U_0, x), x) \tag{H.30}$$

where the constant C multiplies F and so F has been rescaled to reflect this. Thus if $H = 0$ then F's parametrized by U_0 satisfy the generalized Hamilton-Jacobi equations (H.15). Note that the multimomenta are not necessarily constant as in the case of 1-dimensional Hamilton-Jacobi canonical transformation, but are simply divergence free.

Another objective might be to have H independent of P_k^ν: $H = H(U, X)$, so that the DDW equations give $\partial_\nu P_k^\nu \approx -\frac{\partial H}{\partial U^k}(U_0, X)$ and $\partial_\nu U^k \approx \frac{\partial H}{\partial P_k^\nu}(U_0, X)$, so that $U^k \approx U_0^k$ is constant on solutions.

Then the equations are:

$$Cp_i^\mu = \frac{\partial F^\mu}{\partial u^i}(u, U_0, x), \tag{H.31}$$

$$\frac{\partial F^\mu}{\partial U^k}(u, U_0, x) = \frac{\partial x^\mu}{\partial X^\lambda}(x) P_k^\lambda(u, p, x), \tag{H.32}$$

$$Det(\frac{\partial X^\kappa}{\partial x^\lambda})(x) H(U(u, p, x), X(x)) - \frac{\partial F^\mu}{\partial x^\mu}(u, U_0, x) = Ch(u, p, x) \tag{H.33}$$

Note that only the type 1 generating function $F(u, U, X)$ makes sense in the multiphase space setting.

Appendix I

Bibliography

Bibliography

[1] Bruce A. Tulczyjew triples and higher Poisson/Schouten structures on Lie algebroids. *Rept.Math.Phys.*, 66:251–276, 2010. math-ph/0910.1243v5.

[2] Echeverria-Enriquez A. Multivector fields and connections. Setting Lagrangean equations in field theories. *J.Math.Phys.*, 39:4578–4603, 1998. dg-ga/9707001v2.

[3] Fomenko A. *Symplectic Geometry.* Gordon and Breach Publishers, 2nd edition, 1995.

[4] Fuster A. Henneaux M. Maas A. BRST-antifield quantization: a short review. hep-th/0506098v2.

[5] Gray A. Curvature identities for Hermitian and almost Hermitian manifolds. *Tohoku Math. Journ.*, 28:601–612, 1976.

[6] Marsden J. Weinstein A. Reduction of symplectic manifolds with symmetry. *Rep. Math. Phys.*, 5:121–130, 1974.

[7] Rogers A. Gauge fixing and BFV quantization. *Class. Quantum Grav.*, 17:389–397, 2000.

[8] Rogers A. Equivariant BRST quantization and reducible symmetries. *King's College London preprint,*, page 22, 2006. hep-th/0604073.

[9] Rogers A. *Supermanifolds, Theory and Applications.* World Scientific Publishing, 2007.

[10] Vinogradov A. A spectral sequence associated with a non-linear differential equation, and the algebro-geometric foundations of Lagrangian field theory with constraints. *Sov. Math. Dokl.*, 19:144–148, 1978.

[11] Vinogradov A. The c-spectral sequence, lagrangian formalism and conservation laws. i and ii. *J. Math. Anal. Appl.*, 100:1–129, 1984.

[12] Barut A.O. *Electrodynamics and Classical Theory of Fields and Particles.* Dover Publications, New York, 1980.

[13] DeWitt B. *Physical Review Letters*, 12:742, 1964.

[14] DeWitt B. *Physical Review*, 160:1113, 1967.

[15] DeWitt B. *Physical Review*, 162:1195 , 1239, 1967.

[16] DeWitt B. *Supermanifolds.* Cambridge University Press, 1984,1992.

[17] Kostant B. *Graded manifolds, graded Lie theory and prequantization, Differential geometrical methods in mathematical physics : proceedings of the symposium held at the University of Bonn, July 1-4, 1975, pp. 177-306.*

[18] Baez J. Hoffnung A. Rogers C. Categorified symplectic geometry and the classical string. *Commun.Math.Phys.*, 293:701–725, 2010. arXiv:0808.0246.

[19] Caratheodory C. *Variationsrechnung und Partielle Differentialgleichungen erster Ordnung.* Teubner, Leipzig, 1935.

[20] Henneaux M. Teitelboim C. *Quantization of Gauge Systems.* Princeton, 1992.

[21] Rovelli C. General relativity and quantum cosmology. Covariant Hamiltonian formalism for field theory: Hamilton-Jacobi equation on the space G. *Lect.Notes Phys.*, (633):36–62, 2003. gr-qc/0207043.

[22] Krupka D. A geometric theory of ordinary first order variational problems in fibered manifolds, i. Critical sections, 180–206, ii. Invariance, 469–476.

[23] McDuff D. Salamon D. *Introduction to symplectic topology.* Oxford Mathematical Monographs, 1995.

[24] Vey D. Multisymplectic formulation of vielbein gravity. DeDonder-Weyl formulation, Hamiltonian (n-1)-forms. *Class. Quantum Grav.*, 32(095005), 2015.

[25] Cannas da Silva A. *Lectures on Symplectic Geometry.* Springer, 2001.

[26] de Azcárraga J. Izquierdo J. *Lie Groups, Lie Algebras, Cohomology and Some Applications in Physics.* Cambridge monographs of mathematical physics, 1995.

[27] Cartan E. *Lecons sur les Invariants Integraux.* Hermann, Paris.

[28] Gotay M. J. Isenberg J. Marsden J. E. Momentum maps and classical relativistic fields. Part 1: Covariant field theory. page 66, November 1997. physics/9801019.

[29] Landau L. Lifshitz E. *Course of theoretical physics : vol.I Mechanics.* Oxford: Pergamon, 1960.

[30] Witten E. Topological sigma models. *Commun Math. Phys.*, 118:411–449, 1988.

[31] Cantrijn F. Hamiltonian structures on multisymplectic manifolds. *Rend. Sem.Mat Univ. Pol. Torino Geom. Strc. for Phys. Theories*, 54(3):225–236, 1996.

[32] Helein F. Multiphase space and gauge in the theory of variations. *Bull. de l'Acad Polon. des Sci. Serie Sci, Math., Astr. et Phys.*, 22:1219–1225, 1973.

[33] Fradkin E. Vilkoviski G. Quantization of relativistic systems with constraints equivalence of canonical and covariant formalisms in quantum theory of gravitational field. *CERN preprint*, 1997.

[34] Martin G. Darboux theorem for multisymplectic manifolds. *Lett. Math Phys.*, 16:133 138., 1988.

[35] Sardanashvily G. *Gauge Theory in Jet Manifolds.* Hadronic Press, Palm Harbor, Florida., 1993.

[36] Sardanashvily G. Fibre bundles, jet manifolds and Lagrangian theory. Lectures for theoreticians. 2009. math-ph/ 0908.1886.

[37] L. Gray A. Hervella. The sixteen classes of almost Hermitian manifolds and their linear invariants. *Ann. Mat. Pura Appl. (4)*, 123:35–58, 1980.

[38] Forger M. Paufler C. Romer H. Hamiltonian multivector fields and Poisson forms in multisymplectic field theory. *Journal of Mathematical Physics*, 46, 2005.

[39] Goldstein H. *Classical mechanics.* Addison-Wesley Pub. Co.,, 1980.

[40] Paufler C. Romer H. DeDonder-Weyl equations and multisymplectic geometry. *Reports on Mathematical Physics*, 49(2/3):325–334, 2002. arXiv:math-ph/0107019.

[41] Paufler C. Romer H. Geometry of Hamiltonian n-vectorfields in multisymplectic field theory. *J. Geom Phys*, 44:52–69, 2002. math-ph/0102008v3.

[42] Weyl H. Geodesic fields in the calculus of variations. *Ann. Math. (2)*, 36:607–629, 1935.

[43] Kastrup H.A. Canonical theories of Lagrangian dynamical systems in physics. *Physics Rep.*, (101):1–167, 1983.

[44] Kastrup H.A. *Canonical Formalisms - including Hamilton-Jacobi Theories - for Classical Fields*, volume Constraints Theory and Relativistic Dynamics. 1986 Workshop Proceedings. World Scientific Publishing, 1987.

[45] Arnol'd V. I. *Mathematical methods of classical mechanics.* Graduate Texts in Mathematics. Springer, 1989.

[46] Kanatchikov I. Basic structures of the canonical formalism for fields base on the DeDonder-Weyl theory. hep-th/9410238.

[47] Kanatchikov I. On field theoretic generalizations of a Poisson algebra . *Rep. Math. Phys.*, 41(1), 1998.

[48] Kanatchikov I. Towards the Born-Weyl quantization of fields . *Int J of Th Phys.*, 41(1), January 1998. hep-th/9709229.

[49] Kugo T. Ojima I. Local covariant operator formalism of non-abelian gauge theories and quark confinement problem. *Suppl. Progr. Theor. Phys.*, 66:14, 1979.

[50] Tyutin I. Gauge invariance in field theory and statistical physics in operator formalism. *Lebedev Physics Institute preprint*, 39, 1975. arXiv:0812.0580.

[51] Bridges T. J. Multi-symplectic structures and wave propagation. *Math. Proc. Camb. Phil. Soc. 121, 147*, 1997.

[52] Bridges T. Hydon P. Lawson J. Multisymplectic structures and the variational bicomplex. *Math. Proc. Camb. Phil*, 148:159–178, 2010.

[53] Figueroa-O'Farill J. BRST cohomology. www.maths.ed.ac.uk/empg/activities/brst. 2006.

[54] Helein F. Kouneiher J. Finite dimensional Hamiltonian formalism for gauge and field theories. *J.Math.Phys.*, (43):2306–2347, 2002. math-ph/0010036.

[55] Hélein F. Kouneiher J. The notion of observable in the covariant Hamiltonian formalism for the calculus of variations with several variables. *Adv. Theor. Math. Phys. 8*, pages 735–777, 2004. arXiv:math-ph/0401047.

[56] Kijowski J. A finite dimensional canonical frormalism in the classical field theory. *Comm. Math. Phys.*, 30:99–128, 1973.

[57] Kijowski J. Multiphase space and gauge in the theory of variations. *Bull. de l'Acad Polon. des Sci. Serie Sci, Math., Astr. et Phys.*, 22:1219–1225, 1973.

[58] Stasheff J. Homological (ghost) approach to constrained Hamiltonian systems. hep-th/9112002v1.

[59] Stasheff J. Homological reduction of constrained Poisson algebras. *J. Diff. Geom.*, 45:221–240, 1997.

[60] Tate J. Homology of Noetherian rings and local rings. *Illinois Journal of Mathematics*, 1:14–27, 1957.

[61] Woodhouse N. M. J. *Geometric Quantization.* Oxford Mathematical Monographs. Clarendon Press, 2nd edition, 1997.

[62] Stasheff J.D. Constrained Poisson algebras and strong homotopy representations. *Bull. Amer. Math. Soc.*, 19:287–290, 1988.

[63] Szczyrba W. Kijowski J. Multisymplectic manifolds and the geometrical construction of the Poisson brackets in the classical field theory. *Geometrie symplectique et physique mathematique*, 1975.

[64] Fadeev l. Popov V. Feynman diagrams for Yang-Mills fields. *Phys.Lett.*, B25(1):29–30, 1967.

[65] Carroll S. M. *Spacetime and Geometry. An Introduction to General Relativity.* Addison Wesley, 2004. ISBN 0-8053-8732-3.

[66] De Leon M. Marrero J. C. De Diego D. M. A geometric Hamilton-Jacobi theory for classical field theories. 2006. arXiv:0801.1181 [math-ph].

[67] Forger M. Salles M. Hamiltonian vector fields on multiphase spaces of classical field theory.

[68] Gotay M. Momentum maps and classical relativistic fields. part 1: Covariant field theory. *available as preprint: Physics/9801019.*

[69] Marsden J. Pekarsky S. Shkoller S. West M. Variational methods, multisymplectic geometry and continuum mechanics. *J. Geom. Phys.*, 38:253–284, 2001.

[70] Nemeschansky D. Preitschopf C. Weinstein M. A BRST primer. *Annals of Physics*, October 1987.

[71] Nemeschansky N. A BRST primer. *Annals of Physics*, 183:226–268, 1988.

[72] Ashtekar A. Bombelli L. Reula O. *The covariant phase space of asymptotically flat gravitational fields, Mechanics, Analysis and Geometry: 200 Years After Lagrange (M. Francaviglia ed.).* North- Holland, Amsterdam., 1991.

[73] Brink L. Di Vecchia P. Howe P. *Phys. Lett. B*, 65:471, 1976.

[74] Dedecker P. Calcul des variations, formes différentielles et champs géodésiques, en géométrie différentielle. *Colloq. Intern. du CNRS LII*, 1953.

[75] Dubois-Violette M. Michor P. A common generalization of the Frolicher-Nijenhuis bracket and the Schouten bracket for symmetric multi-vector fields . *Indagationes Math. N. S.*, 6:51–66, 1995. alg-geom/9401006v1.

[76] Garcia P. Geometria simplectica en la teoria de campos,. *Collect. Math. 19, 1–2, 73*, 1968.

[77] Woit P. Notes on BRST. 2008. Department of Mathematics, Columbia University.

[78] Woit P. BRST and Dirac Cohomology. 2009. Preliminary Draft Version. http://www.math.columbia.edu/ woit/brstdirac.pdf.

[79] Anco S. Tung R. Covariant Hamiltonian boundary conditions in general relativity for spatially bounded space-time regions. *, J. Math. Phys.*, 43:5531–5566, 2002.

[80] Becchi C. Rouet A. Stora R. Renormalization of gauge theories. *Ann. Phys.*, 98(2):287–321, 1976.

[81] Howe P. Tucker R. *Phys. Lett. A*, 10:L155, 1977.

[82] Goldschmidt H. Sternberg S. The Hamilton–Cartan formalism in the calculus of variations. *Ann. Inst. Fourier Grenoble*, 23,1:203– 267, 1973.

[83] Hrabak S. On a multisymplectic formulation of classical BRST symmetry for first order field theories. Part 1 Algebraic structures. 1999. math-ph/9901012.

[84] Hrabak S. On a multisymplectic formulation of classical BRST symmetry for first order field theories. Part 2 Geometric structures. 1999. math-ph/9901013.

[85] Konstant B. Sternberg S. Symplectic reduction, BRS cohomology and infinite dimensional Clifford algebras. *Ann. Physics*, 176:49–113, 1987.

[86] Kouranbaeva S. Shkoller S. A variational approach to second-order multisymplectic field theory. *J. Geom. Phys.*, 35:333–366, 2000.

[87] Marsden J. Shkoller S. Multisympletic geometry, covariant Hamiltonians, and water waves. *Math. Proc. Camb. Phil. Soc.*, 1998.

[88] Weinberg S. *The Quantum Theory of Fields, vol.1: Foundations.* Cambridge University Press, 1995.

[89] DeDonder T. *Théorie Invariantive du Calcul des Variations.* Gauthier-Villars Paris, 1935.

[90] Goto T. *Prog. Theor. Phys.*, 46:1560, 1971.

[91] Lepage T. Champs stationnaires, champs géodésiques et formes intégrables. *Bull. Acad. Roy. Belg. Cl. Sci.*, 28, 1942.

[92] Marsden J. Ratiu T. *Introduction to mechanics and symmetry.* Springer-Verlag New York, 1999.

[93] Kanatchikov I. V. Canonical structure of classical field theory in the polymomentum phase space. *Rept.Math.Phys. 41*, (41):49–90, 1998. hep-th/9709229.

[94] Kanatchikov I. V. DeDonder-Weyl theory and a hypercomplex extension of quantum mechanics to field theory. *Rept.Math.Phys.*, (43):157, 1999.

[95] Kanatchikov I. V. Precanonical quantization and the Schroedinger wave functional. *Phys Lett*, A283:25, 2001.

[96] Volterra V. Sopra una estensione della teoria Jacobi-Hamilton del calculo delle variazioni. *Rend. Cont. Acad. Lincei.*, 6:127–136, 1890.

[97] Volterra V. Sulle equazioni differenziali che provengono da questiono di calcolo. *Rend. Cont. Acad. Lincei.*, 6:42–54, 1890.

[98] van Holten J.W. Aspects of BRST quantization. *Lect.Notes Phys.*, 659::99–166, 2005. hep-th/0201124v1.

[99] Gribov V.N. Quantization of non-abelian gauge theories. *Nuclear physics B*, 130:1–19, 1978.

[100] Kijowski J. Tulczyjew W. *A symplectic framework for field theories.* Springer-Verlag, 1979.

[101] Tulczyjew W. The Euler-Lagrange resolution. *Lecture Notes in Mathematics. Springer-Verlag, New York*, 836:22–48, 1980.